Algebraic filling inequalities and cohomological width

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Meru Alagalingam

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Universität
Augsburg
University

Gutachter:

1. Prof. Dr. Bernhard Hanke, Universität Augsburg
2. Prof. Mikhail Gromov, Ph.D., Institut des Hautes Études Scientifiques
3. Prof. Larry Guth, Ph.D., Massachusetts Institute of Technology

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Summary

This PhD thesis was supervised by Bernhard Hanke and is profoundly inspired by [Gro09] and [Gro10].

Let $f\colon X \to Y$ be a continuous map. For any $y \in Y$ the topological complexity of the fiber $f^{-1}(y) \subseteq X$ can be measured by the rank of the restriction homomorphism $H^k(X;\mathbb{Z}) \to H^k(f^{-1}(y);\mathbb{Z})$ and a reasonable notion of the complexity of X relative to Y is given by the minimax expression

$$\operatorname{width}_k(X/Y) := \min_{f\colon X\to Y} \max_{y\in Y} \operatorname{rk}\left[H^k(X;\mathbb{Z}) \to H^k(f^{-1}(y);\mathbb{Z})\right] \tag{0.1}$$

called *cohomological width*. I.e. every continuous map $f\colon X \to Y$ admits a point $y \in Y$ such that the fiber $f^{-1}(y) \subseteq X$ satisfies

$$\operatorname{rk}\left[H^k(X;\mathbb{Z}) \to H^k(f^{-1}(y);\mathbb{Z})\right] \geq \operatorname{width}_k(X/Y).$$

It is interesting to evaluate this width for various classes of spaces X and Y or find at least lower bounds for it.

Using *isoperimetric inequalities* in the cohomology algebra $H^*(T^n;\mathbb{Z})$ Gromov could prove

$$\operatorname{width}_k(T^n/\mathbb{R}) \geq \left(1 - \frac{2k}{n}\right)\binom{n}{k}$$

and asked whether this can be generalised from tori to products of higher-dimensional projective spaces. Such a generalisation will be given in Theorem 3.2.4.

We will call the dimension of the target space Y in (0.1) the *codimension* and width problems are considerably more difficult if $\dim Y \geq 2$. Using ideas from Lusternik–Schnirelmann theory it was shown in [Gro09] that for every q-dimensional simplicial complex Y and $n \geq p(q+1)$ we have

$$\operatorname{width}_k(T^n/Y) \geq \binom{p}{k}. \tag{0.2}$$

In [Gro10] the question was raised whether and how one can prove cohomological width inequalities using a certain geometric *filling argument*. In a discussion Larry Guth emphasised the importance of this technique and proposed to investigate *cohomological filling inequalities* which resulted in the crucial Filling Lemma 4.3.2. Using this we could prove the following

Theorem 4.2.1. If N^q is a manifold we have

$$\operatorname{width}_1(T^n/N) = n - q.$$

In Theorem 4.5.2 we generalise Theorem 4.2.1 from tori to arbitrary essential manifolds with free abelian fundamental groups. Furthermore we could use rational homotopy theory in order to prove an algebraic version of a cohomological filling inequality in $(S^p)^n$ implying the following

Theorem 4.6.1. Let $p \geq 3$ be odd, $n \leq p - 2$ and N an orientable q-manifold. We have

$$\mathrm{width}_p((S^p)^n/N) \geq n - q$$

and this holds for any simply connected, closed pn-manifold M with $M \simeq_{\mathbb{Q}} (S^p)^n$.

Theorem 4.2.1 is the first estimate admitting arbitrary codimension q which was proved using a filling argument. Compared to (0.2) the class of target spaces is restricted to q-manifolds but the resulting inequality is stronger. This also holds for Theorem 4.6.1.

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And a very sincere thanks to my family.

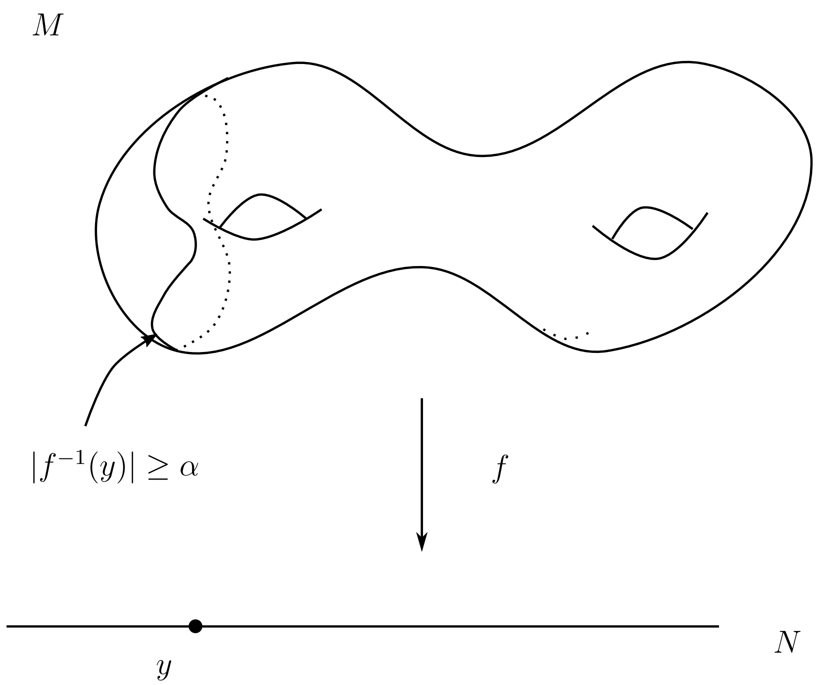
1 Introduction

Much of this introduction is inspired by [Gut14].

Consider two spaces M and N and assume that there is a notion of *size* of subsets of M, i.e. to any subset $A \subseteq M$ we can assign a real number $|A|$. We are interested in lower bounds α such that

$$\sup_{y \in N} |f^{-1}(y)| \geq \alpha$$

for any continuous map $f\colon M \to N$. In other words the bound α shall be *uniform in all continuous maps* $f\colon M \to N$.



Equivalently we are interested in

$$\mathrm{width}(M/N) := \inf_{f\colon M \to N} \sup_{y \in N} |f^{-1}(y)|$$

and lower bounds of this minimax expression where the infimum runs over all continuous maps $f\colon M \to N$. As the picture suggests such inequalities are called *waist* or *width in-*

equalities. Abusing the terminology of topology the preimage $f^{-1}(y)$ of a point $y \in N$ is called the *fiber of* f *over* y.

We will almost exclusively consider the case where M^n and N^q are manifolds. If we furthermore assume that we are in the generic case, i.e. $f\colon M \to N$ is smooth and $y \in N$ is a regular value of f the fiber $f^{-1}(y)$ will be codimension q embedded submanifold in M. Therefore we call q, i.e. the dimension of the target space N, the *codimension* of the waist problem $\mathrm{width}(M/N)$. As we will see in Section 4 compared to the case of codimension $q = 1$ waist inequalities are conceptually harder to prove in codimensions $q \geq 2$.

One main difficulty however when proving waist inequalities is that if we do impose restrictions on the maps $f\colon M \to N$ we can not assume anything about the fibers $f^{-1}(y)$, e.g. whether they are manifolds, connected, simply connected etc. E.g. if M is metrisable any closed subset $A \subseteq M$ is the fiber $f^{-1}(0)$ of the map $f := \mathrm{dist}(A, -)\colon M \to \mathbb{R}$.

1 Waist of the sphere inequality

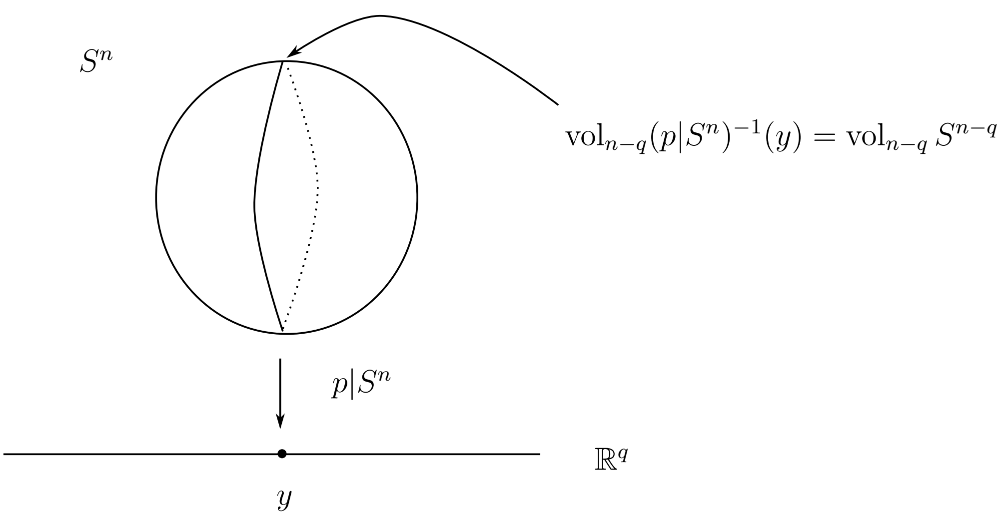
One important example is given by the following

Theorem 1.1 (Waist of the sphere inequality, [Alm65], [Gro83], [Gro03])**.** Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. Any continuous map $f\colon S^n \to \mathbb{R}^q$ admits a point $y \in \mathbb{R}^q$ such that the preimage $f^{-1}(y) \subseteq S^n$ satisfies

$$\mathrm{vol}_{n-q}\, f^{-1}(y) \geq \mathrm{vol}_{n-q}\, S^{n-q} \tag{1.1}$$

where vol_{n-q} denotes the $(n-q)$-dimensional Hausdorff-volume in S^n and $S^{n-q} \subseteq S^n$ is any $(n-q)$-dimensional equator in S^n.

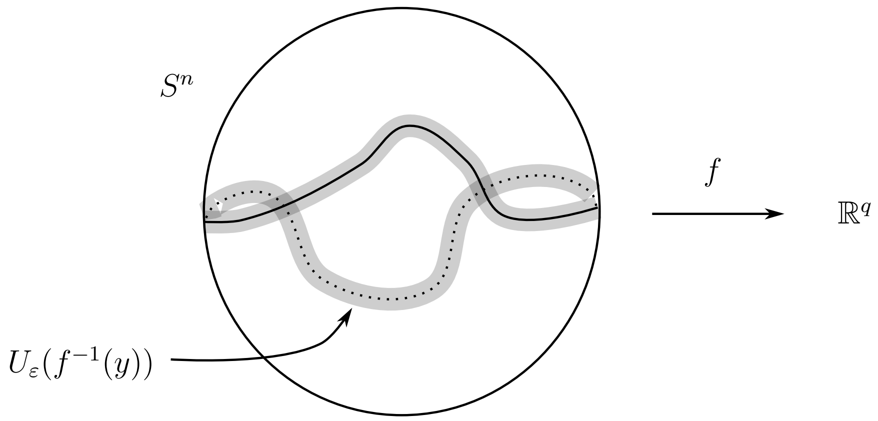
Let $p\colon \mathbb{R}^{n+1} \to \mathbb{R}^q$ be an arbitrary linear projection. The fibers of its restriction $p|S^n\colon S^n \to \mathbb{R}^q$ are $(n-q)$-spheres the largest of which constitutes an $(n-q)$-dimensional equator.



For any subset $A \subseteq S^n$ and $\varepsilon > 0$ let $U_\varepsilon(A)$ denote the open ε-neighbourhood of A in S^n. There is another version of the waist of the sphere inequality:

Theorem 1.2 (Waist of the sphere inequality, ε-neighbourhood version)**.** Let $\varepsilon > 0$. Every continuous map $f\colon S^n \to \mathbb{R}^q$ admits a point $y \in \mathbb{R}^q$ such that

$$\operatorname{vol}_n(U_\varepsilon(f^{-1}(y))) \geq \operatorname{vol}_n(U_\varepsilon S^{n-q}).$$



Note that ε in the theorem above does not have to be small. E.g. we can choose $n = q$ and $\varepsilon = \frac{\pi}{2}$ and get the following

Corollary 1.3. Every continuous map $f\colon S^n \to \mathbb{R}^n$ admits a point $y \in \mathbb{R}^n$ such that the open subset $U_{\frac{\pi}{2}}(f^{-1}(y)) \subseteq S^n$ has full vol_n-measure, i.e.

$$\operatorname{vol}_n U_{\frac{\pi}{2}}(f^{-1}(y)) = \operatorname{vol}_n S^n.$$

Proof. In the case $n = q$ the equator $S^{n-q} = S^0 \subset S^n$ consists of two antipodal points and the open $\frac{\pi}{2}$-neighbourhood $U_{\frac{\pi}{2}}(S^0) = S^n \setminus S^{n-1}$ has full measure with respect to vol_n. □

This can also be seen as a corollary of the Borsuk–Ulam theorem which states that every continuous map $f\colon S^n \to \mathbb{R}^n$ admits to antipodal points $\pm y \in S^n$ satisfying $f(y) = f(-y)$. Nevertheless Corollary 1.3 is not equivalent to the Borsuk–Ulam theorem since the preimage $f^{-1}(y)$ could consist of more than two points.

There also is a non-trivial qualitative version of the waist of the sphere inequality:

Theorem 1.4 (Waist inequality, non-sharp version, [Gro83, p. 134])**.** Let (M^n, g) be a closed Riemannian manifold and $q < n$. There exists a constant $C(M, g, q) > 0$ such that every continuous map $f\colon M \to \mathbb{R}^q$ admits a point $y \in \mathbb{R}^q$ with

$$\operatorname{vol}_{n-q} f^{-1}(y) \geq C_{M,g,q}.$$

Even this non-sharp version has the following strong

Corollary 1.5 (Invariance of domain)**.** For $m \neq n$ there is no homeomorphism $\Phi\colon \mathbb{R}^m \xrightarrow{\cong} \mathbb{R}^n$.

Proof. It suffices to show that there is no continuous injective map $\Phi\colon \mathbb{R}^{q+1} \to \mathbb{R}^q$ for some $q \geq 1$. Assume such a map exists. Consider the projection

$$\begin{aligned} L\colon S^{q+1} &\to \mathbb{R}^{q+1} \\ (x_1, \dots, x_{q+2}) &\mapsto (x_1, \dots, x_{q+1}). \end{aligned}$$

The fibers of L are discrete and consist of at most 2 points. All the fibers of the composition $\Phi \circ L\colon S^{q+1} \to \mathbb{R}^q$ are discrete as well and hence their 1-dimensional Hausdorff volume vanishes. This contradicts the non-sharp version of the waist inequality. □

The connections to the Borsuk–Ulam theorem and the invariance of domain demonstrate that the waist inequality captures a topological phenomenon.

Historically the first proof of Theorem 1.1 was given in [Alm65] and uses deep geometric measure theory. After the short proof of the non-sharp Theorem 1.4 there is another hard proof of the sharp Theorem 1.1 in [Gro03].

2 Further waist inequalities

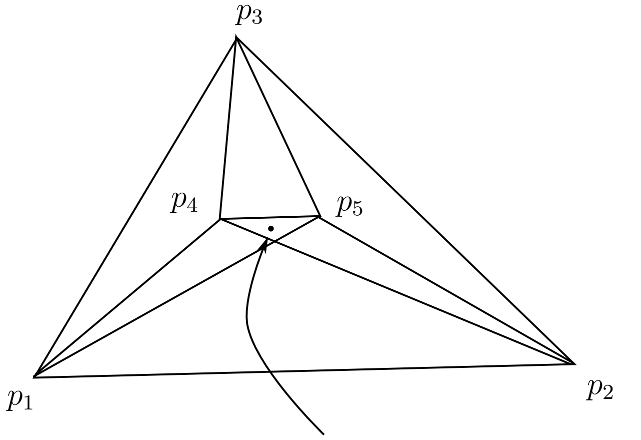
The short proof of the non-sharp waist inequality [Gro83, p. 134] exhibits a general technique called a *filling argument in a space of cycles*. It has already been used to prove waist inequalities of very different flavours two of which we want to present here (cf. Theorems 2.2 and 2.5).

The first one is a result from geometric combinatorics.

Theorem 2.1 ([BF84])**.** Let $P \subset \mathbb{R}^2$ be a set of n points in general position and consider the $\binom{n}{3}$ triangles they define. Then there exists a point $y \in \mathbb{R}^2$ which is contained in at least

$$\frac{2}{9}\binom{n}{3} - O(n^2)$$

of these triangles.



y lies in 5 out of $\binom{10}{3} = 10$ triangles

There is an interpretation of Theorem 2.1 that makes it appear more like a waist inequality. Every n-element subset $P \subset \mathbb{R}^2$ defines an affine linear map $f_P \colon \Delta^{n-1} \to \mathbb{R}^2$. To every subset $A \subseteq \Delta^{n-1}$ assign the volume

$$\operatorname{vol} A := \#\{\text{2-dimensional faces of } \Delta^{n-1} \text{ which intersect } A\}.$$

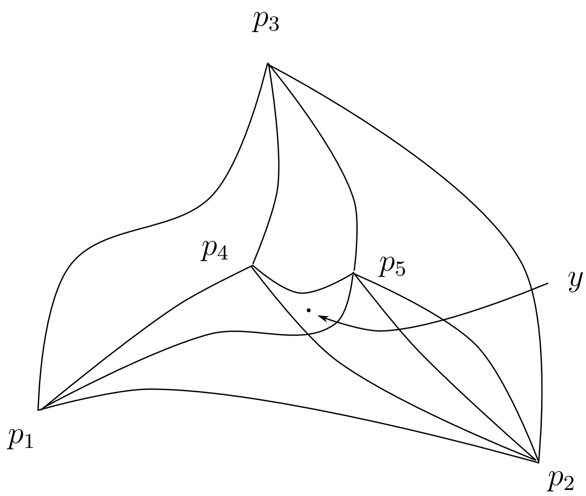
Theorem 2.1 states that for every affine linear map $f \colon \Delta^{n-1} \to \mathbb{R}^2$ there exists a point $y \in \mathbb{R}^2$ with

$$\operatorname{vol} f^{-1}(y) \geq \frac{2}{9}\binom{n}{3} - O(n^2).$$

Using a filling argument in a suitable space of cycles the following generalisation could be proven:

Theorem 2.2 ([Gro10]). Any *continuous* map $f \colon \Delta^{n-1} \to \mathbb{R}^2$ admits a point $y \in \mathbb{R}^2$ such that

$$\operatorname{vol} f^{-1}(y) \geq \frac{2}{9}\binom{n}{3} - O(n^2).$$



y lies in 5 out of $\binom{10}{3} = 10$ triangles

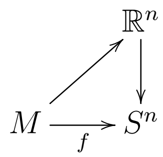
We already saw that the waist of the sphere inequality is related to the Borsuk–Ulam theorem. Consider the following generalisation of the Borsuk–Ulam theorem:

Theorem 2.3 ([Hop44]). Let (M^n, g) be a closed Riemannian manifold and $d > 0$ arbitrary. Every continuous map $f\colon M \to \mathbb{R}^n$ admits two points $x, y \in M$ connected by a geodesic arc of length d such that $f(x) = f(y)$.

The Borsuk–Ulam theorem follows by taking $M^n := S^n$ with the round metric and $d = \pi$. In the same paper Hopf poses the following

Conjecture 2.4. Let (M^n, g) be a closed Riemannian manifold and $d > 0$ arbitrary. Every continuous map $f\colon M \to S^n$ of *degree* 0 admits two points $x, y \in M$ connected by a geodesic arc of length d such that $f(x) = f(y)$.

The philosophy behind this conjecture is that we use to think of a degree 0 map $f\colon M \to S^n$ like a map which is not surjective or a map $\widetilde{f}\colon S^n \to \mathbb{R}^n$. In fact the conjecture is only open for all degree 0 maps $f\colon M \to S^n$ for which there does not exist a factorisation



for an arbitrary map $\mathbb{R}^n \to S^n$. In a footnote Hopf claims that such maps $f\colon S^2 \to S^2$ exist but this is unclear to us.

The following result similar to Conjecture 2.4 was also proven using a filling argument inside a certain space of 0-cycles.

Theorem 2.5 ([AKV12]). Let (M^n, g) be a closed Riemannian manifold with injectivity radius $\rho > 0$ and N^n an open manifold. For any continuous map $f\colon M \to N$ and any $0 < d \leq \rho$ there exist two points $x, y \in M$ connected by a geodesic arc of length d such that $f(x) = f(y)$.

3 Cohomological waist inequalities

Until now we measured the waist

$$\operatorname{width}(M/N) := \inf_{f\colon M \to N} \sup_{y \in N} |f^{-1}(y)|$$

with respect to cardinality or some metric volume $|\cdot|$ on M. In this paper we are rather interested in the *topological complexity* of the preimage sets $f^{-1}(y) \subseteq M$. If the fibers are surfaces or closed manifolds one could consider their genus or simplicial volume.

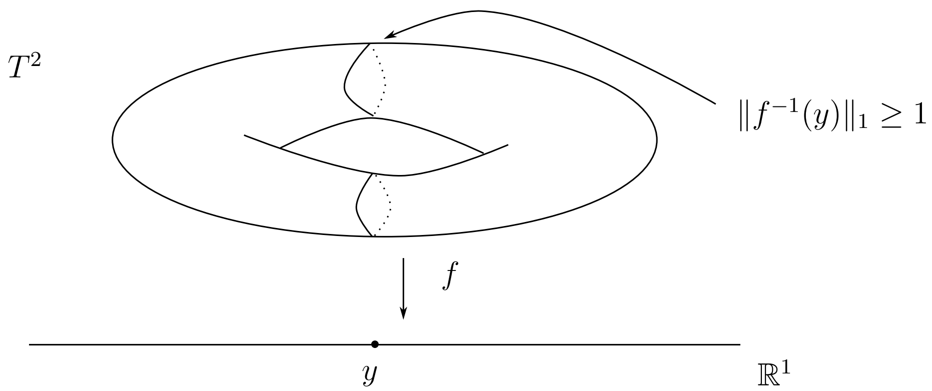
However there are two reasonable ways to measure the topological complexity of subsets $A \subseteq M$ without further assumptions. For the rest of this section let R be a coefficient ring such that the rank of a homomorphism of R-modules makes sense (e.g. $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Q}$). Consider the ranks of the cohomological[1] restriction homomorphisms

$$\|A\|_k := \operatorname{rk}\left[H^k(M;R) \to H^k(A;R)\right] \text{ and}$$
$$\|A\|_* := \operatorname{rk}\left[H^*(M;R) \to H^*(A;R)\right]$$

where $k \geq 0$. These expressions are called the *cohomological volumes* of A and they measure how much the subset $A \subseteq M$ captures of the cohomology of the ambient space M. The waists associated to these cohomological volumes,

$$\operatorname{width}_k(M/N) := \min_{f\colon M\to N} \max_{y\in N} \|f^{-1}(y)\|_k \text{ and}$$
$$\operatorname{width}_*(M/N) := \min_{f\colon M\to N} \max_{y\in N} \|f^{-1}(y)\|_*,$$

are called the *cohomological widths of M over N*.



A simpler definition of cohomological width would be by taking the ranks of the cohomology groups of the fibers themselves instead of the ranks of the restriction homomorphisms, i.e.

$$w_k(M/N) := \min_{f\colon M\to N} \max_{y\in N} \operatorname{rk}\left[H^k(f^{-1}(y);R)\right] \text{ and}$$
$$w_*(M/N) := \min_{f\colon M\to N} \max_{y\in N} \operatorname{rk}\left[H^*(f^{-1}(y);R)\right].$$

We will explain in Remark 2.2.3 (ii) why we refrain from doing so.

Remark 3.1 (The construction is like in [Gro88, Example (H_1'')], also cf. [Gro09, p. 14])**.** Let M and N be simplicial complexes of dimensions n respectively q. For any $k \geq \frac{n+1}{q+1}$ the degree k cohomological width

$$\operatorname{width}_k(M/N) := \min_{f\colon M\to N} \max y \in N \operatorname{rk}\left[H^k(M;R) \to H^k(f^{-1}(y);R)\right] = 0.$$

[1]Technically we want to use Čech cohomology here but we will start to deal with these subtleties in Chapter 2.

Note that we do not claim that all the groups $H^k(f^{-1}(y); R)$ vanish, only that the restriction homomorphisms $H^k(M; R) \to H^k(f^{-1}(y); R)$ do.

A more obvious upper bound for $\|A\|_k$ is given by $\operatorname{rk} H^k(M; R)$ and this yields an upper bound for $\operatorname{width}_k(M/N)$ and analogously for $\|A\|_*$ and width_*. In order to obtain nontrivial lower bounds for cohomological width we should substitute the source manifold S^n from Theorem 1.1 by some manifold M with rich cohomology algebra, such as high-dimensional tori T^n. Essentially there were two known inequalities about cohomological widths (Theorems 3.2 and 3.3).

Theorem 3.2 ([Gro10, pp. 424])**.** For $k < \frac{n}{2}$ we have

$$\operatorname{width}_k(T^n/\mathbb{R}) \geq \left(1 - \frac{2k}{n}\right)\binom{n}{k},$$

i.e. any continuous map $f\colon T^n \to \mathbb{R}$ admits a point $y \in \mathbb{R}$ such that the rank of the restriction homomorphism satisfies

$$\operatorname{rk}\left[H^k(T^n; \mathbb{Z}) \to H^k(f^{-1}(y); \mathbb{Z})\right] \geq \left(1 - \frac{2k}{n}\right)\binom{n}{k}.$$

The second inequality is unfortunately ambiguously called the *maximal fiber inequality.* In order to state it in full generality we need to explain some machinery first. All notions and theorems from the rest of this section have been developed in [Gro09, pp. 13] and [Gro10, Section 4.2].

Let F be a field and $A = \bigoplus_{i=0}^{\infty} A^i$ a graded F-algebra with unity. The product in A shall be denoted by $\smile$ and we assume that it is commutative in the graded sense. For $r \geq 1$ define

$$A^{/r} := \bigcap_{\substack{I \text{ graded ideal} \\ \dim_F(A/I) < r}} I.$$

For $r < s$ we have $A^{/r} \supseteq A^{/s}$. Let $\iota_r\colon A^{/r} \hookrightarrow A$ denote the inclusion and $\mu_d\colon A^{\otimes d} \to A$ the d-fold cup product map. We define $\operatorname{rk}_d^{\smile} A$ as the maximal number r such that the composition

$$\left(A^{/r}\right)^{\otimes d} \xrightarrow{\iota_r^{\otimes d}} A^{\otimes d} \xrightarrow{\mu_d} A$$

does not vanish.

For the rest of this section Y^q is always a metrisable space of Lebesgue covering dimension q, e.g. a q-dimensional simplicial complex. We can now state the maximal fiber inequality.

Theorem 3.3. Let X be compact and assume that $A := H^*(X; F)$ is finitely generated as an F-algebra. Then we have

$$\operatorname{width}_*(X/Y^q) \geq \operatorname{rk}_{q+1}^{\smile}(A).$$

It is not difficult to see that if $X \simeq X_1 \times \ldots \times X_k$ for closed connected oriented manifolds X_i we have

$$\mathrm{rk}_k^{\smile} H^*(X; F) \geq \min_{i=1}^{k} \dim_F \left(H^*(X_i; F)\right).$$

In particular for tori T^n of dimensions $n \geq p(q+1)$ we have

$$\mathrm{rk}_{q+1}^{\smile} H^*(T^n; F) \geq 2^p$$

from which we get the following

Corollary 3.4. For $n \geq p(q+1)$ we have

$$\mathrm{width}_*(T^n/Y^q) \geq 2^p,$$

i.e. every continuous map $T^n \to Y^q$ admits a point $y \in Y$ such that

$$\mathrm{rk}\left[H^*(T^n; F) \to H^*(f^{-1}(y); F)\right] \geq 2^p.$$

A careful analysis of the proof of Theorem 3.3 shows that this $y \in Y$ actually satisfies

$$\mathrm{rk}\left[H^p(T^n; F) \to H^p(f^{-1}(y); F)\right] \geq 1$$

and this purely algebraically implies (similar to Motivation 4.3.1)

$$\mathrm{rk}\left[H^k(T^n; F) \to H^k(f^{-1}(y); F)\right] \geq \binom{p}{k} \tag{3.1}$$

for every $0 \leq k \leq p$.

We can compare the different lower bounds, e.g. for $\mathrm{width}_k(T^{2p}/\mathbb{R})$. Theorem 3.2 yields

$$\mathrm{width}_k(T^{2p}/\mathbb{R}) \geq \left(1 - \frac{k}{p}\right)\binom{2p}{k} \tag{3.2}$$

whereas we get from Theorem 3.3 that

$$\mathrm{width}_k(T^{2p}/\mathbb{R}) \geq \binom{p}{k}. \tag{3.3}$$

The bound (3.2) is significantly stronger than (3.3) but the latter holds for all 1-dimensional target spaces Y^1, not just $Y^1 = \mathbb{R}$.

Theorem 3.2 is a codimension 1 result and its proof uses so-called *isoperimetric inequalities in algebras.* We will explain this technique in Chapter 3. In [Gro10, p. 509] it was asked whether and how this inequality can be generalised to products of higher-dimensional projective spaces. We will prove such a generalisation in Theorem 3.2.4. Both Theorem 3.2 and Theorem 3.2.4 hold for all closed manifolds with the correct cohomology algebra (cf. Remark

3.2.19).

Theorem 3.3 on the other hand is a result admitting target spaces Y^q of arbitrary codimension $q \geq 1$. Its proof is far less geometric and uses Lusternik–Schnirelmann type argument. This argument and isoperimetric inequalities in algebras are the only known techniques to prove cohomological waist inequalities.

In Theorem 3.3 we did not require X to be a manifold and it holds for all compact spaces X with cohomology algebra isomorphic to A. This is strange since the problem of finding cohomological waist inequalities is by no means homotopy invariant, e.g. there is no real reason why two homotopy equivalent spaces such as T^n and $T^n \times I$ should satisfy

$$\mathrm{width}_k(T^n/\mathbb{R}) = \mathrm{width}_k(T^n \times I/\mathbb{R})$$

although the lower bounds given by Theorem 3.3 are the same.

This indicates that Theorem 3.3 and e.g. the estimate (3.1) for $\mathrm{width}_k(T^n/Y^q)$ are far from sharp and do not capture all of the geometry of the source manifold T^n and the target space Y^q and whether the latter is a manifold or not. Using a filling argument in a space of $(n-q)$-cycles in T^n we sharpen the bound from Corollary (3.1) as follows.

Theorem 4.2.1. If N^q is a manifold we have

$$\mathrm{width}_1(T^n/N) = n - q,$$

i.e. for every continuous $f\colon T^n \to N$ there exists a point $y \in N$ such that

$$\mathrm{rk}\left[H^1(T^n;\mathbb{Z}) \to H^1(f^{-1}(y);\mathbb{Z})\right] \geq n - q$$

and one can construct maps f such that equality holds.

It is the first non-trivial sharp evaluation of cohomological width, slightly improves the best known lower bound for $\mathrm{width}_1(T^n/\mathbb{R})$ coming from Theorem 3.2 and generalises to arbitrary source manifolds that need not be tori but can be arbitrary essential m-manifolds with fundamental group $\mathbb{Z}^n$ (cf. Theorem 4.5.2). Using rational homotopy theory we could also prove the following estimate about cartesian powers of higher-dimensional spheres.

Theorem 4.6.1. Let $p \geq 3$ be odd and $n \leq p - 2$. Consider $M = (S^p)^n$ or any simply connected, closed manifold of dimension pn with the rational homotopy type $(S^p)^n_{\mathbb{Q}}$. For any orientable manifold N^q we have

$$\mathrm{width}_p(M/N) \geq n - q.$$

Theorems 4.2.1 and 4.6.1 are the first cohomological waist inequalities admitting arbitrary codimensions $q \geq 1$ that have been proven using a filling argument. Theorem 4.6.1 is the first lower bound on width_p with $p > 1$ that has been proven using this technique.

4 Filling argument

In this section we want to give two proofs of the waist of the sphere inequality (cf. Theorem 1.1). When proving such theorems we want to avoid focussing on the regularity of the map f or the preimage subsets $f^{-1}(y)$. Therefore we restrict ourselves to the analysis of the following statement:

For every *generic smooth* map $f\colon S^n \to \mathbb{R}^q$ there exists a *regular value* $y \in \mathbb{R}^q$ satisfying

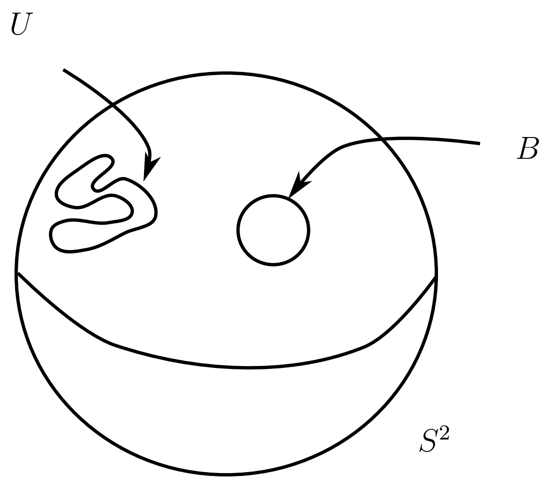
$$\operatorname{vol}_{n-q} f^{-1}(y) \geq \operatorname{vol}_{n-q} S^{n-q}. \tag{4.1}$$

This is convenient since in this case $f^{-1}(y)$ is an $(n-q)$-dimensional embedded submanifold in S^n. Note that we cannot drop the (admittedly vague) genericity condition since e.g. a constant map $f\colon S^n \to \mathbb{R}^q$ does not admit a regular value $y \in \mathbb{R}^q$ satisfying (4.1). In Section 2.2 we will explain rigorously what we mean by genericity and why we can always assume maps to be generic in the context of cohomological waist inequalities.

In the case of codimension $q = 1$ the waist of the sphere inequality essentially follows from the following

Theorem 4.1 (Isoperimetric inequality in S^n)**.** Let $U \subseteq S^n$ be open. For every round ball B satisfying $\operatorname{vol}_n B = \operatorname{vol}_n U$ we have

$$\operatorname{vol}_{n-1} \partial U \geq \operatorname{vol}_{n-1} \partial B.$$

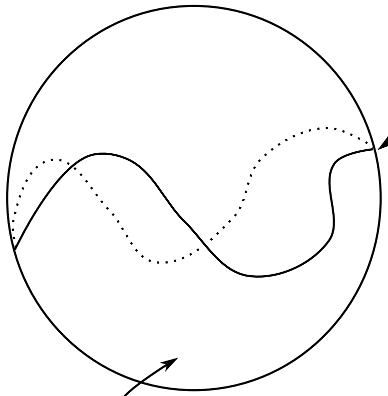


With this isoperimetric inequality we can give a simple proof of the waist of the sphere inequality in codimension $q = 1$. Consider a map $f\colon S^n \to \mathbb{R}$. If we assume that f is generic, e.g. a Morse function, the mapping

$$\begin{aligned} \mathbb{R} &\to [0, \operatorname{vol}_n S^n] \\ y &\mapsto \operatorname{vol}_n f^{-1}(-\infty, y) \end{aligned}$$

is continuous. By the mean value theorem there exists a point $y \in \mathbb{R}$ such that

$$\mathrm{vol}_n f^{-1}(-\infty, y) = \frac{1}{2} \mathrm{vol}_n S^n.$$



The boundary of $f^{-1}(-\infty, y)$ is $f^{-1}(y)$ and we want to apply the isoperimetric inequality above. Since $\mathrm{vol}_n f^{-1}(-\infty, y) = \frac{1}{2} \mathrm{vol}_n S^n$ we can choose any hemisphere B as a round comparison ball. The boundary of a hemisphere is an $(n-1)$-dimensional equator and we get

$$\mathrm{vol}_{n-1} f^{-1}(y) = \mathrm{vol}_{n-1} \partial f^{-1}(-\infty, y) \geq \mathrm{vol}_{n-1} \partial B = \mathrm{vol}_{n-1} S^{n-1}.$$

Remark 4.2. The philosophy behind this proof is that codimension 1 waist inequalities can be always proven using isoperimetric inequalities. We will follow this strategy in Chapter 3 where we develop the notion of isoperimetric inequalities in algebras in order to prove cohomological waist inequalities.

Nevertheless it not clear how to generalise this proof in order to prove the waist inequality in arbitrary codimensions $q \geq 1$ which we recall for convenience

Theorem 1.4 (Waist inequality, non-sharp version, [Gro83, p. 134])**.** Let (M^n, g) be a closed Riemannian manifold and $q < n$. There exists a constant $C(M, g, q) > 0$ such that every continuous map $f\colon M \to \mathbb{R}^q$ admits a point $y \in \mathbb{R}^q$ with

$$\mathrm{vol}_{n-q} f^{-1}(y) \geq C_{M,g,q}.$$

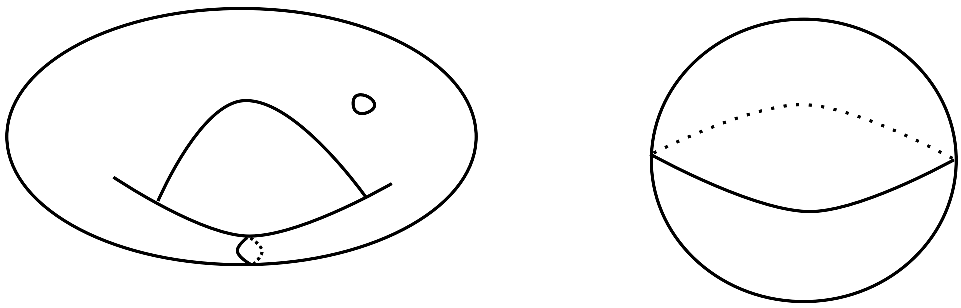
Recall that even this non-sharp version is non-trivial as it implies the invariance of domain. The core ingredient for the proof is the following

Lemma 4.3 (Filling inequality, [Gro83, Sublemma 3.4.B'])**.** Let (M^n, g) be a compact Riemannian manifold and $1 \leq k \leq n$. There is a small constant $\alpha = \alpha_{M^n,g,k} > 0$, a large constant $C = C_{M^n,g,k}$ such that for every cycle $y \in C_k(M)$ with $\operatorname{vol}_k y \leq \alpha$ there exists a chain $z \in C_{k+1}(M)$ satisfying

$$\partial z = y \text{ and } \operatorname{vol}_{k+1} z \leq C\left(\operatorname{vol}_k y\right)^{\frac{k+1}{k}}.$$

The coefficients of all chains can be chosen in $\mathbb{Z}$ or $\mathbb{Z}_2$.

Example 4.4. For the sphere we can choose $\alpha_1 = \infty$ since every 1-cycle can be filled. For the torus α_1 has to smaller than the shortes non-contractible loop. In both cases α_2 can be chosen as any number smaller than the volume of the surfaces.

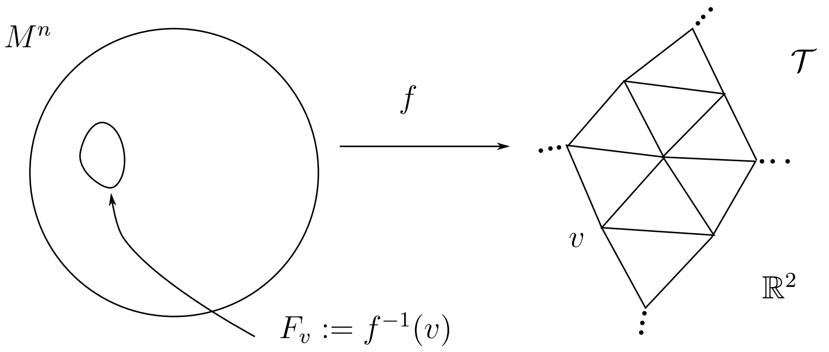


This Filling inequality can be proven by isometrically embedding M^n into some $\mathbb{R}^N$ and use the known filling inequalities there. Note that this is a non-sharp generalisation of the isoperimetric inequality in S^n. The chain $y \in C_k(M)$ in the Filling Lemma really has to be a cycle as a chain with non-empty boundary cannot be filled regardless of any volume assumptions. The Filling Lemma also shows that every non-zero homology class $[y] \in H_k(M; Z)$ captures a positive amount of k-volume. This motivates the definition of the k-*systole* of a Riemannian manifold (cf. [Gro83]).

Proof sketch of Theorem 1.4, [Gro83, p. 134]. The main idea of the proof is already contained in the case of codimension $q = 2$ to which we restrict ourselves. We proceed by contradiction and assume that for every $\varepsilon > 0$ there exists a generic smooth map $f\colon M^n \to \mathbb{R}^2$ such that the fiber over every point $y \in \mathbb{R}^2$ satisfies $\operatorname{vol}_{n-2} f^{-1}(y) < \varepsilon$. As we will have to choose ε smaller and smaller and hence change the choice of f we introduce the Landau O notation

$$\operatorname{vol}_{n-2} f^{-1}(y) \in O(\varepsilon)$$

by which we mean that we can make the expression $\operatorname{vol}_{n-2} f^{-1}(y)$ arbitrarily small as we let $\varepsilon \to 0$.



Choose a smooth triangulation $\mathcal{T}$ of the target $\mathbb{R}^2$ which is *generic and fine*. By a generic triangulation we mean that all vertices of the triangulation are regular values of f, i.e. the preimages of vertices $F_v := f^{-1}(v)$ are codimension 2 submanifolds, the preimages of edges $F_{[v,w]} := f^{-1}[v,w]$ are codimension 1 submanifolds with boundary etc. In all our pictures the preimages of points are 1-manifolds hence they depict the case $n-q=1$ or equivalently $n=3$. For every simplex σ of $\mathcal{T}$ let $F_\sigma := f^{-1}(\sigma)$. By a fine triangulation we mean that all preimages have small volume, i.e. for every k-simplex σ we have

$$\mathrm{vol}_{n-2+k} F_\sigma \in O(\varepsilon).$$

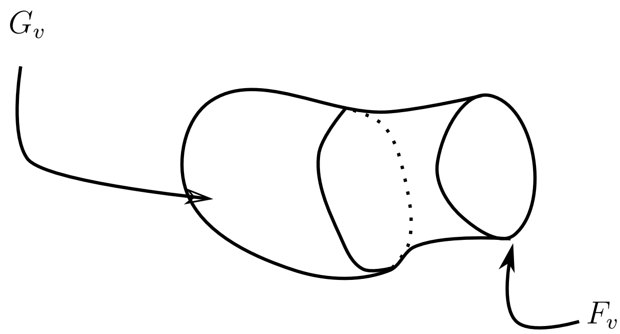
Intuitively this means that these volumes can be made arbitrarily small if ε is chosen sufficiently small. For the 0-simplices we have this by assumption and for the higher-dimensional simplices we can achieve this by barycentric subdivision.

For every k-simplex σ of $\mathcal{T}$ the preimage F_σ is a manifold with corners and these (non-canonically) define chains in $C_{n-2+k}(M)$. In this proof all chains are meant with coefficients in $\mathbb{Z}_2$ and we do not denote the difference between the subset $F_\sigma \subseteq M$ and the chain $F_\sigma \in C_{n-2+k}(M)$.

For every vertex v of $\mathcal{T}$ we have $\mathrm{vol}_{n-2} F_v \in O(\varepsilon)$ and can apply the Filling Lemma to F_v and obtain a filling $G_v \in C_{n-1}(M)$ (i.e. $\partial G_v = F_v$) with

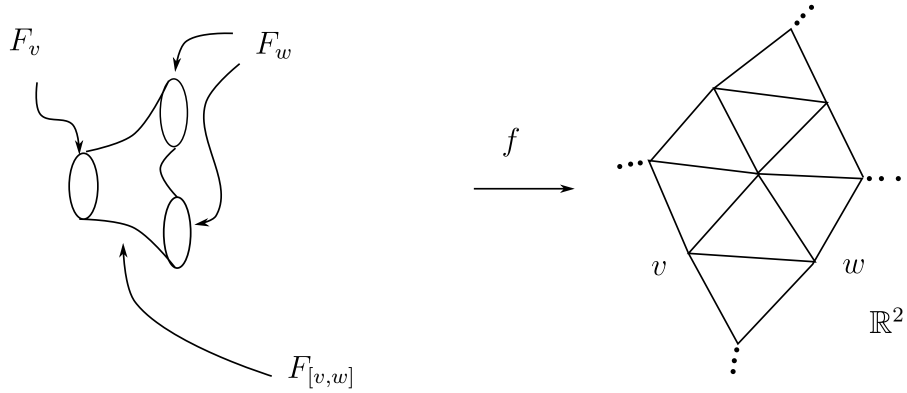
$$\mathrm{vol}_{n-1} G_v \leq C \left(\mathrm{vol}_{n-2} F_v\right)^{\frac{n-1}{n-2}}.$$

In particular we have $\mathrm{vol}_{n-1} G_v \in O(\varepsilon)$.



For every edge $[v, w]$ of $\mathcal{T}$ the preimage $F_{[v,w]}$ is a manifold with boundary $f^{-1}\{v, w\}$ and the corresponding chain has boundary

$$\partial F_{[v,w]} = F_v + F_w.$$

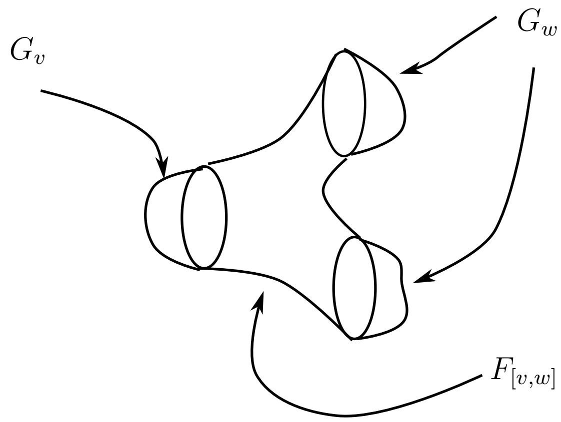


In the singular chain complex $C_\bullet(M^n; \mathbb{Z}_2)$ we have

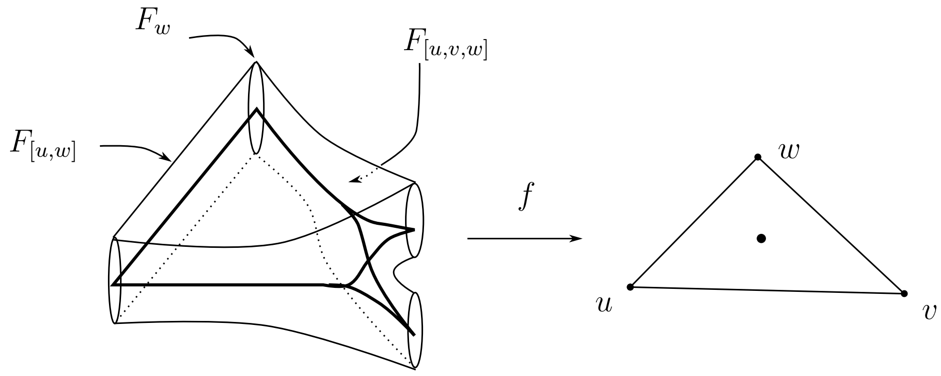
$$\partial(G_v + F_{[v,w]} + G_w) = F_v + (F_v + F_w) + F_w = 0$$

and since $\mathrm{vol}_{n-1}(G_v + F_{[v,w]} + G_w) \in O(\varepsilon)$ we can apply the Filling Lemma to obtain $G_{[v,w]} \in C_n(M; \mathbb{Z}_2)$ such that $\partial G_{[v,w]} = G_v + F_{[v,w]} + G_w$ and

$$\mathrm{vol}_n(G_v + F_{[v,w]} + G_w) \in O(\varepsilon).$$



We continue this process and consider the preimages of triangles $F_{[u,v,w]}$. In the example picture below $F_{[u,w]}$ is a cylinder and both $F_{[u,v]}$ and $F_{[v,w]}$ are pairs of pants. The preimage $F_{[u,v,w]}$ is a solid double torus. The bold line is mapped to the barycentre of $[u, v, w]$ and the farther a point of $F_{[u,v,w]}$ is from this core line the closer it is mapped to $\partial[u, v, w]$.



We have $\partial(F_{[u,v,w]} + G_{[u,v]} + G_{[v,w]} + G_{[u,w]}) = 0$ and

$$\operatorname{vol}_n(F_{[u,v,w]} + G_{[u,v]} + G_{[v,w]} + G_{[u,w]}) \in O(\varepsilon)$$

and using the Filling Lemma we get a chain $G_{[u,v,w]} \in C_{n+1}(M; \mathbb{Z}_2)$ satisfying

$$\partial G_{[u,v,w]} = F_{[u,v,w]} + G_{[u,v]} + G_{[v,w]} + G_{[u,w]} \tag{4.2}$$

(and $\operatorname{vol}_{n+1} G_{[u,v,w,]} \in O(\varepsilon)$). Summing up this equation (4.2) over all triangles of $\mathcal{T}$ we get

$$\partial \sum_{[u,v,w]} G_{[u,v,w]} = \sum_{[u,v,w]} F_{[u,v,w]} + G_{[u,v]} + G_{[v,w]} + G_{[u,w]} \tag{4.3}$$

$$= \sum_{[u,v,w]} F_{[u,v,w]} \tag{4.4}$$

The sum $\sum_{[u,v,w]} F_{[u,v,w]}$ is an n-chain in M^n and its support is all of M. In fact it is not difficult to see that $\sum_{[u,v,w]} F_{[u,v,w]}$ represents the fundamental class in $H_n(M^n; \mathbb{Z}_2)$ (for details about this observation cf. Proposition 4.1.8) But equation (4.3) contradicts the fact that the fundamental classes can not be written as boundaries. □

In order to prove cohomological waist inequalities in higher codimensions Larry Guth suggested to imitate the proof scheme above but all the volumes of cycles have to be measured in the following new sense[2].

Definition 4.5 (Cohomological volume)**.** Let $C_*(X)$ denote the singular chain complex of a topological space X. Every chain $c \in C_p(X)$ is a formal linear combination $c = \sum_{i=1}^n z_i \sigma_i$ of singular p-simplices σ_i in X. The *support* $\operatorname{supp} c \subseteq X$ *of* c is the union of all $\operatorname{im} \sigma_i$ for which the coefficient z_i does not vanish. Let $\iota_c\colon \operatorname{supp} c \hookrightarrow X$ denote the inclusion. The *cohomological* k*-volume of* c is defined as

$$|c|_k := \operatorname{rk} H^k \iota_c.$$

[2]Of course one could also try to find other topological waist inequalities by proving filling inequalities for other measures of topological complexity, e.g. genus of surfaces or simplicial volume of closed manifolds.

Of course one needs a replacement for the Filling Lemma. E.g. in order to find a lower bound for $\mathrm{width}_*(T^n/\mathbb{R}^2)$ in a first step one has to show that for every cycle $c \in C_{n-2}(T^n)$ with $\partial c = 0$ there exists a filling, i.e. a chain $d \in C_{n-1}(T^n)$ satisfying $\partial d = c$, such that the cohomological volume $|d|_1$ can be controlled in terms of $|c|_1$.

Instances of such cohomological filling inequalities are the isoperimetric inequalities from Chapter 3, the Filling Lemma 4.3.2 for tori and the Rational Filling Lemma 4.6.11 for cartesian powers of higher-dimensional spheres. In Chapter 5 we propose a rigorous and general definition of a cohomological filling inequality encompassing these examples. Although greatly inspired by the proof scheme above and the suggested modification all of the remaining chapters, especially Chapter 3 and 4, are formally independent of this chapter.

2 Cohomological width and ideal valued measures

In this chapter we introduce the notion of ideal valued measures which is a useful tool to study cohomological width.

We will use two cohomology theories, namely Čech cohomology and simplicial cohomology and it should always be clear from the context which one we mean depending on whether we evaluate it on topological spaces or simplicial complexes. Nevertheless and in order to avoid confusion we will consistently denote Čech cohomology by $\check{H}^*$ and simplicial cohomology simply by H^*. For the rest of this paper let R be a coefficient ring which may be arbitrary unless specified otherwise.

1 Ideal valued measures

Definition 1.1 (Cohomological width)**.** Let R be a coefficient ring such that the rank of a homomorphism between R-modules makes sense, e.g. $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Q}$.

(i) For every continuous map $f\colon X \to Y$ the expressions

$$\begin{aligned} \operatorname{width}_*(f) &:= \max_{y\in Y} \operatorname{rk}\left[\check{H}^*X \to \check{H}^* f^{-1}y\right] \text{ and} \\ \operatorname{width}_k(f) &:= \max_{y\in Y} \operatorname{rk}\left[\check{H}^kX \to \check{H}^k f^{-1}y\right] \end{aligned}$$

are called the *total* or *degree k cohomological width of f*.

(ii) For fixed topological spaces X and Y the minima

$$\begin{aligned} \operatorname{width}_*(X/Y) &:= \min_{f\in C(X,Y)} \operatorname{width}_*(f) \text{ and} \\ \operatorname{width}_k(X/Y) &:= \min_{f\in C(X,Y)} \operatorname{width}_k(f) \end{aligned}$$

where $C(X, Y)$ denotes the set of all continuous maps $f\colon X \to Y$ are called the *total* or *degree k cohomological width of X over Y*.

We are interested in lower bounds of $\operatorname{width}_1(X/Y)$ for fixed manifolds X and Y. The following important observation motivates the rest of this section.

For every continuous map $f\colon X \to Y$ and every $y \in Y$ we have

$$\operatorname{rk}\left[\check{H}^*X \to \check{H}^*f^{-1}y\right] = \operatorname{rk}\left[\check{H}^*X \big/ \ker\left[\check{H}^*X \to \check{H}^*f^{-1}y\right]\right].$$

Therefore we want to systematically study kernels of restriction homomorphisms $\check{H}^*X \to \check{H}^*C$ for closed subsets $C \subseteq X$ and this motivates the following

Definition 1.2 (Standard ideal valued measure, pushforward)**.** Let X be a topological space and let $A := \check{H}^*(X)$. Let τ_X denote the system of all open subsets of X and $\mathcal{I}(A)$ the set of all graded ideals $I \subseteq A$.

(i) Assign to every open $U \subseteq X$ a graded ideal $\mu_X(U) \subseteq A$ via

$$\begin{aligned} \mu_X\colon \tau_X &\to \mathcal{I}(A) \\ U &\mapsto \ker\left[\check{H}^*X \to \check{H}^*(X \setminus U)\right]. \end{aligned}$$

This map μ_X is called *the standard*[1] *ideal valued measure on* X (or $\mathcal{I}(A)$*-valued measure* if one wants to emphasise the ambient algebra). It trivially satisfies $\mu_X(\emptyset) = 0$ (normalisation), $\mu_X(U_1) \subseteq \mu_X(U_2)$ whenever $U_1 \subseteq U_2$ (monotonicity) and $\mu_X(X) = A$ (fullness).

(ii) For any continuous map $f\colon X \to Y$ the assignment

$$\begin{aligned} f_*\mu_X\colon \tau_Y &\to \mathcal{I}(A) \\ U &\mapsto \mu_X(f^{-1}U) = \ker\left[\check{H}^*X \to \check{H}^*(X \setminus f^{-1}U)\right] \end{aligned}$$

is called the *pushforward of* μ_X *along* f. It also satisfies normalisation, monotonicity and fullness. Note that $f_*\mu_X$ maps open subsets of Y to ideals in $\check{H}^*(X)$.

Corollary 1.3. For every continuous map $f\colon X \to Y$ and every closed subset $C \subseteq Y$ we have

$$\begin{aligned} \operatorname{rk}\left[\check{H}^*X \to \check{H}^*f^{-1}C\right] &= \operatorname{rk}\left[\check{H}^*X \big/ \ker\left[\check{H}^*X \to \check{H}^*f^{-1}C\right]\right] \\ &= \operatorname{rk}\left[\check{H}^*X \big/ f_*\mu_X(Y \setminus C)\right] \text{ and similarly} \end{aligned}$$

$$\operatorname{rk}\left[\check{H}^kX \to \check{H}^kf^{-1}C\right] = \operatorname{rk}\left[\check{H}^kX \big/ f_*\mu_X(Y \setminus C) \cap \check{H}^kX\right].$$

Using specific features of Čech cohomology we want to derive more properties of μ_X and $f_*\mu_X$.

[1]Later on we will give an abstract definition of an ideal valued measure and if X is a compact manifold μ_X will be an instance of it.

Proposition 1.4 (Additivity)**.** Let X be a manifold and $A := \check{H}^*X$ its Čech cohomology algebra. The standard $\mathcal{I}(A)$-valued measure μ_X on X satisfies *additivity*, i.e. for any two disjoint, open $U_1, U_2 \subseteq X$ we have

$$\mu_X(U_1 \dot\cup U_2) = \mu(U_1) + \mu(U_2).$$

The analogous statement also holds for the pushforward measure $f_*\mu_X$ along any continuous map $f\colon X \to Y$.

Repetition 1.5 (Čech cohomology, [ES52])**.** Let X be a topological space. For any open cover $\alpha = (U_i)_{i\in I}$ of X the *nerve of* α is a simplicial complex X_α with one vertex for every index $i \in I$ satisfying $U_i \neq \emptyset$ and an n-simplex $\{i_0, \ldots, i_n\} \subset I$ belongs to X_α iff $U_{i_0} \cap \ldots \cap U_{i_n} \neq \emptyset$. If α and β are two open coverings we write $\alpha < \beta$ if β refines α and this relation turns the set of open covers into a directed set. One can show that whenever $\alpha < \beta$ there is a well-defined map between simplicial cohomology groups $H^qX_\alpha \to H^qX_\beta$. The *Čech cohomology groups* are defined as the direct limit

$$\check{H}^q(X) := \varinjlim_{\alpha} H^q(X_\alpha).$$

Every element in this direct limit can be represented by a cohomology class $z \in H^q(X_\alpha)$ for a sufficiently fine open cover α of X. Čech cohomology satisfies the Eilenberg–Steenrod axioms (cf. [Dow50]) and from this we get the following

Theorem 1.6 (Comparison between Čech and singular cohomology)**.** For every CW pair (X, A) there is an isomorphism

$$\eta_{(X,A)}\colon \check{H}^*(X, A; R) \to H^*(X, A; R)$$

which defines a natural equivalence $\eta\colon \check{H}^* \to H^*$ of functors from CW pairs to R-modules.

The proof of Proposition 1.4 needs some preparation.

Lemma 1.7. Let X be a topological space, $V \subseteq X$ closed and $[z] \in \check{H}^qX$ such that $[z]|V = 0 \in \check{H}^qV$. Then there exists an open cover δ of X such that $[z]$ can be represented by some cohomology class $[z]_\delta \in H^qX_\delta$ and the restriction homomorphism satisfies

$$\begin{aligned} H^qX_\delta &\to H^qV_{\delta|V} \\ [z]_\delta &\mapsto 0 \end{aligned}$$

where $\delta|V$ denotes the induced open cover on V.

Proof. Every cohomology class $[z] \in \check{H}^qX = \varinjlim_\alpha H^qX_\alpha$ can represented by some cohomology class $[z]_\alpha \in H^q(X_\alpha)$ for some sufficiently fine open cover α of X. Consider its image under the restriction homomorphism

$$\begin{aligned} H^q X_\alpha &\to H^q V_{\alpha|V} \\ [z]_\alpha &\mapsto [z]_\alpha | \, (\alpha|V)\,. \end{aligned}$$

By assumption this restricted class vanishes in $\check{H}^q V = \varinjlim_\beta H^q V_\beta$, i.e. there exists a refinement γ of $\alpha|V$ such that the restriction satisfies

$$\begin{aligned} H^q V_{\alpha|V} &\to H^q V_\gamma \\ [z]_\alpha | \, (\alpha|V) &\mapsto 0. \end{aligned}$$

Since $V \subseteq X$ is closed we can extend γ to an open cover $\widetilde{\gamma}$ of X. Let δ be a common refinement of α and $\widetilde{\gamma}$, in particular $\delta|V$ refines γ.

On this level of the direct system we can represent $[z] \in \check{H}^q X$ by an element $[z]_\delta \in H^q X_\delta$ and its image under the restriction homomorphism of simplicial cohomology satisfies

$$H^q X_\delta \to H^q V_{\delta|V} \tag{1.1}$$
$$[z]_\delta \mapsto 0. \tag{1.2}$$

□

Lemma 1.8. Let (K, L) be a pair of simplicial complexes and $z \in C^q K$ a simplicial cocycle such that $[z]|L = 0 \in H^q L$ vanishes. Then there is a cochain $\widetilde{z} \in C^q K$ which is cohomologous to z (i.e. $[z] = [\widetilde{z}]$ in $H^q K$) such that $\widetilde{z}|L = 0$ in $C^q L$.

Proof. There exists a cochain $w \in C^{q-1} L$ such that $z|L = dw$. We can extend this cochain to a $(q-1)$-cochain $\widetilde{w}$ on K by setting

$$\widetilde{w}(\sigma) := \begin{cases} w(\sigma) & \text{if } \sigma \in L_{q-1} \\ 0 & \text{otherwise.} \end{cases}$$

For $\widetilde{z} := z - d\widetilde{w}$ and an arbitrary $\sigma \in L_q$ we have

$$\widetilde{z}(\sigma) = (z - d\widetilde{w})(\sigma) = z(\sigma) - d\widetilde{w}(\sigma) = z(\sigma) - \widetilde{w}(\partial\sigma) = z(\sigma) - w(\partial\sigma) = z(\sigma) - dw(\sigma) = 0.$$

□

Lemma 1.9. Let K be a simplicial complex and L_1 and L_2 two subcomplexes of K such that $K = L_1 \cup L_2$. Let $[z] \in H^q K$ such that $[z]|L_1 \cap L_2 = 0 \in H^q(L_1 \cap L_2)$. Then there exists a decomposition $[z] = [z_1] + [z_2]$ for cohomology classes $[z_i] \in H^q K$ satisfying $z_i|L_i = 0 \in C^q L_i$ ($i = 1, 2$).

Proof. Using the preceding lemma we can assume that $z|L_1 \cap L_2 = 0 \in C^q(L_1 \cap L_2)$. For any $\sigma \in K_q$ define $z_1 \in C^q K$ by

$$z_1(\sigma) := \begin{cases} z(\sigma) & \text{if } \sigma \in (L_2)_q \\ 0 & \text{if } \sigma \in (L_1)_q. \end{cases}$$

and $z_2 \in C^q K$ analogously. Since $z|L_1 \cap L_2 = 0 \in C^q(L_1 \cap L_2)$ these cochains are well-defined and satisfy $z = z_1 + z_2$. They are cocycles since for every $\sigma \in (L_1)_{q+1}$ we have

$$dz_1(\sigma) = z_1(\partial\sigma) = 0$$

and for every $\sigma \in (L_2)_{q+1}$ we get

$$dz_1(\sigma) = z_1(\partial\sigma) = z(\partial\sigma) = dz(\sigma) = 0.$$

□

Definition 1.10 (Good cover)**.** An open cover $(U_i)_{i\in I}$ of a topological space X is called *good* iff all finite intersections $U_{i_0} \cap \ldots \cap U_{i_k}$ are either empty or contractible.

Remark 1.11. (i) If X is a manifold it has a good cover given by geodesically convex balls with respect to an arbitrary Riemannian metric [BT95, Theorem 5.1]. This also shows that every open cover of X can be refined by a good one.

(ii) In the following proof we will only use that all finite intersections $U_{i_0} \cap \ldots \cap U_{i_k}$ are connected but for manifolds X this is not substantially easier than to prove that they are contractible.

Lemma 1.12. Let X be a topological space and $V_1, V_2 \subseteq X$ closed subsets covering X. For any open cover $\alpha = (U_i)_{i\in I}$ of X the corresponding nerves satisfy

$$X_\alpha = (V_1)_{\alpha|V_1} \cup (V_2)_{\alpha|V_2}.$$

Moreover we have

$$(V_1 \cap V_2)_{\alpha|V_1\cap V_2} \subseteq (V_1)_{\alpha|V_1} \cap (V_2)_{\alpha|V_2} \tag{1.3}$$

and if α is a good cover we have equality in (1.3).

Proof. Any q-simplex of X_α corresponds to a finite nonempty intersection $U_{i_0} \cap \ldots \cap U_{i_q} \neq \emptyset$ and this yields a nonempty intersection $U_{i_0} \cap \ldots \cap U_{i_q} \cap V_1 \neq \emptyset$ or $U_{i_0} \cap \ldots \cap U_{i_q} \cap V_2 \neq \emptyset$ proving the first equality.

Every q-simplex of $(V_1 \cap V_2)_\alpha$ corresponds to a nonempty intersection $U_{i_0} \cap \ldots \cap U_{i_q} \cap V_1 \cap V_2 \neq \emptyset$. Hence both $U_{i_0} \cap \ldots \cap U_{i_q} \cap V_1$ and $U_{i_0} \cap \ldots \cap U_{i_q} \cap V_2$ are nonempty proving the inclusion (1.3).

Let α be a good cover and assume there exists a q-simplex $\sigma = \{i_0, \dots, i_q\}$ in

$$((V_1)_\alpha \cap (V_2)_\alpha) \setminus (V_1 \cap V_2)_\alpha$$

and denote $U_\sigma = U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset$. Since α is good U_σ is contractible hence connected.

Since σ is not a simplex of $(V_1 \cap V_2)_\alpha$ we have $U_\sigma \cap V_1 \cap V_2 = \emptyset$. Because it is a simplex of both $(V_1)_\alpha$ and $(V_2)_\alpha$ we have $U_\sigma \cap V_j \neq \emptyset$ ($j = 1, 2$). Thus the $U_\sigma \cap V_j$ ($j = 1, 2$) constitute a decomposition of U_σ into closed, disjoint and nonempty subsets. This contradicts the connectedness of U_σ. □

Proof of Proposition 1.4. Let us prove the additivity of $f_*\mu_X$ assuming we have already proven that of μ_X. The preimages $f^{-1}U_1$ and $f^{-1}U_2$ are still disjoint and open and the additivity of μ_X implies

$$f_*\mu_X(U_1 \dot\cup U_2) = \mu_X(f^{-1}U_1 \dot\cup f^{-1}U_2) = \mu_X(f^{-1}U_1) + \mu_X(f^{-1}U_2) = f_*\mu_X(U_1) + f_*\mu_X(U_2).$$

So it remains to prove the additivity of μ_X.

Consider the complements $V_i := X \setminus U_i$ ($i = 1, 2$). These are closed and cover X. The only inclusion which does not follow from monotonicity is

$$\mu(U_1 \dot\cup U_2) \subseteq \mu(U_1) + \mu(U_2).$$

Let $[z] \in \mu_X(U_1 \dot\cup U_2) \subseteq \check{H}^q X$, i.e. $[z]$ is a Čech cohomology class such that

$$[z]|V_1 \cap V_2 = 0 \in \check{H}^q(V_1 \cap V_2).$$

We have to show that $[z]$ can be written as a sum $[z] = [z_1] + [z_2]$ ($[z_i] \in \check{H}^q X$) such that $[z_i]|V_i = 0$ ($i = 1, 2$).

By Lemma 1.7 we can assume that there exists an open cover δ of X such that $[z] \in \check{H}^q X = \varinjlim_\alpha H^q X_\alpha$ can be represented by a cohomology class $[z]_\delta \in H^q X_\delta$ and $[z]_\delta| \left(V_1 \cap V_2\right)_{\delta|V_1 \cap V_2} = 0 \in H^q \left(V_1 \cap V_2\right)_{\delta|V_1 \cap V_2}$. Using Remark 1.11 we can further refine δ and assume that it is a good cover. Now we got rid of the direct systems and their limits and we can restrict ourselves to the case of ordinary, simplicial cohomology groups.

Consider the subcomplexes $L_i := (V_i)_{\delta|V_i} \subseteq X_\delta$ ($i = 1, 2$). By Lemma 1.12 we have $X_\delta = L_1 \cup L_2$ and

$$(V_1 \cap V_2)_{\delta|V_1 \cap V_2} = L_1 \cap L_2. \tag{1.4}$$

With this notation (1.1) becomes $[z]_\delta|L_1 \cap L_2 = 0 \in H^q(L_1 \cap L_2)$. Using Lemma 1.9 there exist $[z_i] \in H^q X_\delta$ ($i = 1, 2$) with $[z] = [z_1] + [z_2]$ and $[z_i]|L_i = 0 \in H^q L_i$ ($i = 1, 2$). The classes $[z_i] \in H^q X_\delta$ descend to the desired elements in $\check{H}^q X$. □

Remark 1.13. (i) One reason why we always insist on using Čech instead of singular cohomology is that we do not know whether additivity in this generality holds for the latter.

(ii) We also do not know if X really needs to be a manifold. Otherwise we could not refine the open cover δ such that it becomes good. We would not have equality (but a proper inclusion) in equation (1.4) and we could not apply Lemma 1.9.

Proposition 1.14 (Multiplicativity)**.** Let X be topological space and $(A, \smile) := (\check{H}^*X, \smile)$ its Čech cohomology algebra where $\smile$ denotes the cup product. The standard $\mathcal{I}(A)$-valued measure μ_X on X satisfies *multiplicativity*, i.e. for any two open subsets $U_1, U_2 \subseteq X$ we have

$$\mu_X(U_1) \smile \mu_X(U_2) \subseteq \mu_X(U_1 \cap U_2) \tag{1.5}$$

where the left hand side is meant as the product of ideals.

The analogous statement also holds for the pushforward measure $f_*\mu_X$ along any continuous map $f\colon X \to Y$.

Proof. Let us prove the multiplicativity of $f_*\mu_X$ assuming we have already proven that of μ_X. The preimages $f^{-1}U_1$ and $f^{-1}U_2$ are still open and the multiplicativity of μ_X implies

$$\begin{aligned} f_*\mu_X(U_1) \smile f_*\mu_X(U_2) &= \mu_X(f^{-1}U_1) \smile \mu_X(f^{-1}U_2) \\ \subseteq \mu_X(f^{-1}U_1 \cap f^{-1}U_2) &= \mu_X(f^{-1}(U_1 \cap U_2)) = f_*\mu_X(U_1 \cap U_2). \end{aligned}$$

So it remains to prove the multiplicativity of μ_X.

Consider the complements $V_i := X \setminus U_i$ $(j = 1, 2)$. The left hand side of (1.5) is additively generated by products of the form $[x_1] \smile [x_2]$ where $[x_i] \in \mu_X(U_i) \subseteq \check{H}^*X$, i.e. $[x_i]|V_i = 0 \in \check{H}^{q_i}V_i$ $(i = 1, 2)$. Using Lemma 1.7 there exists an open cover δ of X such that each $[x_i] \in \check{H}^{q_i}X = \varinjlim_\alpha H^{q_i}X_\alpha$ is represented by some cohomology class $[x_i]_\delta \in H^{q_i}X_\delta$ and for $i = 1, 2$ the restriction homomorphisms satisfy

$$\begin{aligned} H^{q_i}X_\delta &\to H^{q_i}(V_i)_{\delta|V_i} \\ [x_i]_\delta &\mapsto 0. \end{aligned}$$

Using Lemma 1.8 we can assume that the classes $[x_i]_\delta$ are represented by simplicial q_i-cocycles $(x_i)_\delta \in C^{q_i}X_\delta$ satisfying $(x_i)_\delta|(V_i)_{\delta|V_i} = 0 \in C^{q_i}(V_i)_{\delta|V_i}$. Let $q := q_1 + q_2$.

Now in this simple case the cup product $[x_1] \smile [x_2] \in \check{H}^qX$ in Čech cohomology is represented by $[x_1]_\delta \smile [x_2]_\delta \in H^qX_\delta$ as well as the restriction $[x_1] \smile [x_2]|V_1 \cup V_2 \in \check{H}^q(V_1 \cup V_2)$ is represented by $[x_1]_\delta \smile [x_2]_\delta|(V_1 \cup V_2)_{\delta|V_1 \cup V_2} \in H^q(V_1 \cup V_2)_\delta$ and we have

$$[x_1]_\delta \smile [x_2]_\delta|\,(V_1 \cup V_2)_{\delta|V_1 \cup V_2} \tag{1.6}$$

$$= [(x_1)_\delta \smile (x_2)_\delta]\,|\,(V_1 \cup V_2)_{\delta|V_1 \cup V_2} \tag{1.7}$$

$$= \left[(x_1)_\delta \smile (x_2)_\delta \middle| (V_1 \cup V_2)_{\delta|V_1 \cup V_2}\right]. \tag{1.8}$$

From Lemma 1.12 we know that

$$(V_1 \cup V_2)_{\delta|V_1 \cup V_2} = (V_1)_{\delta|V_1} \cup (V_2)_{\delta|V_2}. \tag{1.9}$$

Let $\delta = (W_i)_{i\in I}$ and fix a strict total ordering on the index set I. As usual one constructs the simplicial chain complex and dual to it the simplicial cochain complex. We need this in order to define the simplicial cup product. Let $\sigma := (v_0 < \cdots < v_q)$ be an arbitrary q-simplex in $(V_1 \cup V_2)_{\delta|V_1\cup V_2}$. Without loss of generality (cf. (1.9)) it is a simplex of $(V_1)_{\delta|V_1}$. The simplicial cup product is defined by

$$(x_1)_\delta \smile (x_2)_\delta(\sigma) = (x_1)_\delta(v_0 < \cdots < v_{q_1}) \cdot (x_2)_\delta(v_{q_1+1} < \cdots < v_{q_1+q_2})$$

and the first factor vanishes because of $(x_1)_\delta|(V_1)_{\delta|V_1} = 0$. This proves that the expression (1.8) vanishes and hence $[x_1] \smile [x_2]|V_1 \cup V_2 = 0 \in \check{H}^q(V_1 \cup V_2)$. □

Proposition 1.15 (Continuity)**.** Let X be a compact topological space and $A := \check{H}^*X$ its Čech cohomology algebra. The standard ideal valued measure μ_X on X satisfies *continuity*, i.e. for any increasing nested sequence of open subsets $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots \subseteq X$ we have

$$\mu\left(\bigcup_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} \mu(U_i). \tag{1.10}$$

The analogous statement also holds for the pushforward measure $f_*\mu_X$ along any continuous map $f\colon X \to Y$.

We will reduce this proposition to the so-called *continuity* of Čech cohomology. In order to state this property properly we need a little preparation.

Definition 1.16. A *compact pair* (X, A) is a pair of spaces such that X is compact and $A \subseteq X$ is closed. In particular A itself is compact. Let Z be a topological space. A sequence of pairs $(X_i, A_i) \subseteq (Z, Z)$ $(i \in \mathbb{N})$ together with inclusions $\iota_i^j\colon (X_j, A_j) \hookrightarrow (X_i, A_i)$ whenever $i < j$ is called a *nested sequence of pairs in* Z and we denote it by $\big((X_i, A_i)_{i\in\mathbb{N}}, \iota_i^j\big)$. For such a nested sequence its *intersection* is the topological pair $(X, A) \subseteq (Z, Z)$ defined by $X := \bigcap_i X_i$ and $A := \bigcap_i A_i$.

We will only need the following very weak version of continuity.

Theorem 1.17 (Continuity of Čech cohomology, [ES52, Theorem 2.6])**.** Let (X, A) be the intersection of a nested sequence of compact pairs. Let $\iota_i\colon (X, A) \hookrightarrow (X_i, A_i)$ denote the inclusion. Each $u \in \check{H}^q(X, A)$ is of the form $\iota_i^* u_i$ for some $i \in \mathbb{N}$ and some $u_i \in \check{H}^q(X_i, A_i)$.

Proof of Proposition 1.15. Let us prove the continuity of $f_*\mu_X$ assuming we have proven that of μ_X. The subsets $(f^{-1}U_i)_{i\in\mathbb{N}}$ form an increasing nested sequence of open subsets of X and the continuity of μ_X implies

$$f_*\mu_X\left(\bigcup_{i=1}^{\infty} U_i\right) = \mu_X\left(f^{-1}\bigcup_{i=1}^{\infty} U_i\right) = \mu_X\left(\bigcup_{i=1}^{\infty} f^{-1}U_i\right) = \bigcup_{i=1}^{\infty} \mu_X(f^{-1}U_i) = \bigcup_{i=1}^{\infty} f_*\mu_X(U_i).$$

It remains to prove the continuity of μ_X.

Consider the complements $V_i := X \setminus U_i$ ($i \in \mathbb{N}$) and $V = \bigcap_i V_i = X \setminus \bigcup_i U_i$. The only inclusion of (1.10) not following from monotonicity is

$$\mu\left(\bigcup_{i=1}^{\infty} U_i\right) \subseteq \bigcup_{i=1}^{\infty} \mu(U_i),$$

i.e. given a cohomology class $[x] \in \check{H}^q X$ satisfying $[z]|V = 0 \in \check{H}^q V$ we have to show the existence of an index $i \in \mathbb{N}$ such that $[z]|V_i = 0$.

Consider the nested sequence of compact pairs given by $(X, V_i)_{i \in \mathbb{N}}$. The intersection of this nested sequence is precisely (X, V). For every $i \in \mathbb{N}$ naturality of the long exact sequence yields the following commutative diagram.

$$\begin{array}{ccccc} \check{H}^q(X, V_i) & \longrightarrow & \check{H}^q X & \longrightarrow & \check{H}^q V_i \\ \downarrow & & \| & & \downarrow \\ \check{H}^q(X, V) & \longrightarrow & \check{H}^q X & \longrightarrow & \check{H}^q V \end{array}$$

In the diagram above every arrow is given by restriction. Because the class $[z] \in \check{H}^q X$ satisfies $[z]|V = 0 \in \check{H}^q V$ we can lift $[z]$ to a class $[\tilde{z}] \in \check{H}^q(X, V)$. By Theorem 1.17 there exists an index $i \in \mathbb{N}$ and a class $[u_i] \in \check{H}^q(X, V_i)$ such that $[u_i]|(X, V) = [\tilde{z}]$. We get $[u_i]|X = [z]$ and the top horizontal exact sequence yields $[z]|V_i = 0$. □

Remark 1.18. (i) The continuity axiom fails if X is not compact. Let $X = B^2 \setminus 0$ and $V_i := \left\{x \in X \middle| x_1 \leq \frac{1}{i}\right\}$. The intersection $\bigcap_i V_i = \{x \in X | x_1 \leq 0\}$ is contractible but the generator of $\check{H}^1 X$ survives when restricted to any V_i.

(ii) Continuity is the second reason why we prefer Čech over singular cohomology.

This motivates the following

Definition 1.19 (Ideal valued measures, [Gro10, Section 4.1])**.** Let Y be a topological space, τ_Y the system of open subsets of Y, $A = \bigoplus_{n=0}^{\infty} A^n$ a graded commutative R-algebra and $\mathcal{I}(A)$ the set of graded ideals $I \subseteq A$. An $\mathcal{I}(A)$*-valued measure* μ *on* Y is a map

$$\mu \colon \tau_Y \to \mathcal{I}(A)$$

assigning a graded ideal $\mu(U) \subseteq A$ to any open $U \subseteq Y$ such that the following properties hold:

(i) Normalisation: $\mu(\emptyset) = 0$.

(ii) Monotonicity: For $U_1 \subseteq U_2$ we have $\mu(U_1) \subseteq \mu(U_2)$.

(iii) Continuity: For any increasing nested sequence $U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots$ we have

$$\mu\left(\bigcup_{i=1}^{\infty} U_i\right) = \bigcup_{i=1}^{\infty} \mu(U_i).$$

(iv) Additivity: For two disjoint open subsets we have

$$\mu(U_1 \dot{\cup} U_2) = \mu(U_1) + \mu(U_2).$$

(v) Multiplicativity: We have

$$\mu(U_1) \cdot \mu(U_2) \subseteq \mu(U_1 \cap U_2)$$

for any open $U_1, U_2 \subseteq Y$.

(vi) Fullness: We have $\mu(Y) = A$.

The main instances of the definition above are given in the following

Corollary 1.20. Let X be a compact manifold and $A := \check{H}^* X$ its Čech cohomology algebra. The standard $\mathcal{I}(A)$-valued measure as well as the pushforward measure $f_* \mu_X$ along any continuous map $f\colon X \to Y$ are ideal valued measures in the sense of the preceding definition.

Proof. This is an immediate consequence of Definition 1.2, Propositions 1.4, 1.14 and 1.15. □

Remark 1.21. (i) Given an ideal valued measure μ on Y it will turn out to be useful to define the *vanishing ideal*

$$\mathbf{0}_\mu(A) := \mu(X \setminus A)$$

for any closed $A \subseteq Y$. If the measure is clear from the context, mostly the standard measure or its pushforward along a continuous map, we will only write $\mathbf{0}(A)$.

(ii) In [Gro10] there is one more axiom. By definition an ideal valued measure satisfies the *intersection property* iff for any two open $U_1, U_2 \subseteq Y$ covering Y we have $\mu(U_1 \cap U_2) = \mu(U_1) \cap \mu(U_2)$. One can show that the standard measure and every pushforward of it satisfy this intersection property but we will not need this for our applications.

2 Genericity

Let X and Y be topological spaces and R a coefficient ring such that the rank of a homomorphism between R-modules makes sense, e.g. $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Q}$.

Definition 2.1 (Waist functionals)**.** Recall from Definition 1.1 that for every continuous map $f\colon X \to Y$ the *total* or *degree k cohomological width of f* is given by

$$\operatorname{width}_*(f) := \max_{y\in Y} \operatorname{rk}\left[\check{H}^*X \to \check{H}^* f^{-1}y\right] \text{ and}$$
$$\operatorname{width}_k(f) := \max_{y\in Y} \operatorname{rk}\left[\check{H}^kX \to \check{H}^k f^{-1}y\right].$$

They give rise to the *waist functionals* width_* and width_k both of which are (not necessarily in any sense continuous) maps $C(X,Y) \to \mathbb{N}_0$ where $C(X,Y)$ is the space of all continuous maps $f\colon X \to Y$.

In the next chapters we will give lower bounds of $\operatorname{width}_1(X/Y)$ for fixed manifolds X and Y. The proofs of these at first only work for generic maps $f\colon X \to Y$, e.g. we will find a lower bound of $\operatorname{width}_1(f)$ for all smooth f which intersect some smooth triangulation of Y transversally. In this section we will show that the same lower bound will also hold for all continuous f. In other words it is sufficient to prove waist inequalities just for (in some sense) generic maps. This is motivated by a sentence in [Gro10, p. 417] about a quantity which "may only *increase* under uniform limits of maps". The aim of this section is to render this precise and give a self-contained proof of the following proposition. We do not know whether it has already been discussed in existing literature.

Proposition 2.2 (Upper semi-continuity of waists)**.** Let X and Y be compact and Y metrisable. If the Čech cohomology algebra $\check{H}^*X$ is finite dimensional the waist functionals width_* and $\operatorname{width}_k\colon C(X,Y) \to \mathbb{N}_0$ are upper semi-continuous with respect to the compact-open topology.

Proof. We will just show the upper semi-continuity of width_*. The corresponding statement for width_k can be proven analogously. Endow Y with an arbitrary metric d. The compact-open topology is identical with the metric topology induced from the uniform norm. As $C(X,Y)$ is a metric space semi-continuity is equivalent to sequential semi-continuity. So given a sequence of functions $f_n\colon X \to Y$ uniformly converging to f we need to show that $\operatorname{width}_*(f_n) \geq \alpha$ for every n implies

$$\operatorname{width}_*(f) \geq \alpha.$$

Hence for every n there exists a point $y_n \in Y$ such that

$$\operatorname{rk}\left[\check{H}^*X \to \check{H}^* f_n^{-1}y_n\right] = \operatorname{rk}\left[\check{H}^*X \big/ (f_n)_*\mu_X(Y \setminus y_n)\right] \geq \alpha$$

where $(f_n)_*\mu_X$ is the pushforward of the standard ideal valued measure on X and we used Corollary 1.3.

Since Y is sequentially compact we can pass to a subsequence and assume that the y_n converge to some point $y \in Y$ and that the convergences $f_n \to f$ and $y_n \to y$ are controlled by

$$d(y_n, y) < \frac{1}{8n^2}$$
$$d(f_n, f) < \frac{1}{8n^2}.$$

We claim the following equality of subsets of X.

$$\bigcup_{n>0} \left\{ x \in X \middle| d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} = \{f(-) \neq y\} \tag{2.1}$$

Let us first discuss the inclusion "$\subseteq$": For every $x \in X$ with $d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n}$ for some n the reverse triangle inequality implies

$$\begin{aligned} d(f(x), y) &\geq d(f_n(x), y_n) - d(f_n(x), f(x)) - d(y_n, y) \\ &> \frac{1}{4n^2} + \frac{1}{n} - \frac{1}{8n^2} - \frac{1}{8n^2} = \frac{1}{n} > 0 \end{aligned}$$

Similarly the inclusion "$\supseteq$" can be shown as follows: If $x \in X$ satisfies $d(f_n(x), y_n) \leq \frac{1}{4n^2} + \frac{1}{n}$ for every n we can conclude

$$\begin{aligned} d(f(x), y) &\leq d(f(x), f_n(x)) + d(f_n(x), y_n) + d(y_n, y) \\ &< \frac{1}{8n^2} + \frac{1}{4n^2} + \frac{1}{n} + \frac{1}{8n^2} \to 0 \end{aligned}$$

and hence $f(x) = y$. This proves (2.1).

Moreover we claim that the sets on the left hand side of (2.1) are nested, i.e. we have

$$\left\{ x \in X \middle| d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} \subseteq \left\{ x \in X \middle| d(f_{n+1}(x), y_{n+1}) > \frac{1}{4(n+1)^2} + \frac{1}{n+1} \right\}. \tag{2.2}$$

If $x \in X$ is an element of the left hand side we have

$$\begin{aligned} d(f_{n+1}(x), y_{n+1}) &> d(f_n(x), y_n) - d(f_n(x), f_{n+1}(x)) - d(y_n, y_{n+1}) \\ &> \frac{1}{4n^2} + \frac{1}{n} - \frac{1}{4n^2} - \frac{1}{4n^2} = \frac{1}{n} - \frac{1}{4n^2} > \frac{1}{4(n+1)^2} + \frac{1}{n+1}. \end{aligned}$$

proving (2.2).

The continuity axiom (which holds by Proposition 1.15 since X is compact) implies

$$\bigcup_{n>0} \mu_X \left\{ x \in X \middle| d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} = \mu_X \{x \in X | f(x) \neq y\}.$$

The left hand side is an increasing sequence of ideals in $\check{H}^*X$ and since the latter is finite dimensional there exists an $n > 0$ such that

$$f_*\mu_X \left\{ x \in X \middle| d(f_n(x), y_n) > \frac{1}{4n^2} + \frac{1}{n} \right\} = f_*\mu_X \{x \in X | f(x) \neq y\}.$$

Monotonicity yields

$$f_*\mu_X \{x \in X | f_n(x) \neq y_n\} \supseteq f_*\mu_X \{x \in X | f(x) \neq y\}$$

proving

$$\mathrm{rk} \left[\check{H}^* \middle/ f_*\mu_X(Y \setminus y) \right] \geq \mathrm{rk} \left[\check{H}^*X \middle/ (f_n)_*\mu_X(Y \setminus y_n) \right] \geq \alpha. \qquad \square$$

Remark 2.3. (i) The waist functionals fail to be lower semi-continuous. Consider the inclusion of the boundary $g\colon S^2 \hookrightarrow D^3$ and the sequence $f_n\colon S^2 \hookrightarrow D^3$ shrinking g to a point, e.g. $f_n(x) = g(x)/n$. This sequence uniformly converges to the constant map f with value $0 \in D^3$ but $\mathrm{width}_2(f_n) = 0$ whereas $\mathrm{width}_2(f) = 1$.

(ii) One question which immediately arises about the definition of cohomological width of a map $f\colon X \to Y$ is why we defined it as

$$\mathrm{width}_k(f) = \max_{y \in Y} \mathrm{rk} \left[H^k X \to H^k f^{-1} y \right]$$

where we could have equally been interested in

$$w_k(f) := \max_{y \in Y} \mathrm{rk}\, H^k f^{-1} y.$$

However this functional $w_k\colon C(X,Y) \to \mathbb{N}_0$ fails to be upper semi-continuous since in the example sequence above we have $w_1(f_n) = 1$ but the limit map satisfies $w_1(f) = 0$.

Nevertheless we clearly have $w_k(f) \geq \mathrm{width}_k(f)$ so any lower bound for $\mathrm{width}_k(f)$ is also one for $w_k(f)$.

(iii) The proof of Proposition 2.2 still works if one weakens the assumption that $\check{H}^*X$ is finite dimensional to $\check{H}^*X$ being finitely generated as an algebra since all finitely generated graded commutative algebras are Noetherian.

3 Codimension 1

Recall the following codimension 1 waist inequality for tori from Chapter 1.

Theorem 1.3.2 ([Gro10, pp. 424])**.** Let $k < \frac{n}{2}$. Any continuous map $f\colon T^n \to \mathbb{R}$ admits a point $y \in \mathbb{R}$ such that the rank of the restriction homomorphism satisfies

$$\operatorname{rk}\left[H^k(T^n;\mathbb{Z}) \to H^k(f^{-1}(y);\mathbb{Z})\right] \geq \left(1 - \frac{2k}{n}\right)\binom{n}{k}.$$

In [Gro10, p. 509] it was asked whether and how this inequality can be generalised to products of higher-dimensional projective spaces. Theorem 1.3.2 is proven using almost only observations about the cohomology algebra $A := H^*(T^n;\mathbb{Z})$ called *isoperimetric inequalities in the algebra* A. We will recap this proof scheme and prove Theorem 2.4 which generalises Theorem 1.3.2 to products of projective spaces.

Let F be an arbitrary base field. Later on we will restrict ourselves to the case $F = \mathbb{Z}_2$. By an algebra we mean a graded unital finite dimensional F-algebra which is commutative in the graded sense.

1 Separation and isoperimetric profiles

For the rest of this section let $A = \bigoplus_{i=0}^{n} A^i$ be an algebra and $0 \leq k \leq n$ a fixed degree. For every finite dimensional F-vector space I let $|I| := \dim_F I$. For any subset $I \subseteq A$ the *orthogonal complement of* I is defined as the linear subspace

$$I^\perp := \{a \in A | \forall i \in I\colon i \cdot a = 0\}.$$

Definition 1.1 (Separation and isoperimetric profiles, [Gro10, pp. 500])**.** For every $0 \leq m \leq |A^k|$ let

$$\mathcal{M}_{A,k}(m) := \max_{\substack{I \subseteq A^k \\ |I| = m}} \left|I^\perp \cap A^k\right|$$

where the maximum runs over all m-dimensional linear subspaces $I \subseteq A^k$. The function $\mathcal{M}_{A,k}$ is called the *separation profile of* A *in degree* k.

For a linear subspace $I \subseteq A^k$ the *algebraic boundary in degree* k is defined as the quotient

$$\partial^k I := A^k \Big/ (I + I^\perp) \cap A^k.$$

The *isoperimetric profile in degree k* $\mathcal{N}_{A,k}$ is defined as the function

$$\mathcal{N}_{A,k}(m) := \min_{\substack{I \subseteq A^k \\ |I|=m}} \left|\partial^k I\right|.$$

If the algebra A and the degree k are clear from the context we will suppress this from the notation.

Remark 1.2. An obvious lower bound for the isoperimetric profile is

$$\mathcal{N}_{A,k}(m) \geq \left|A^k\right| - m - \mathcal{M}_{A,k}(m).$$

The geometric importance of these purely algebraic concepts lies in the following

Proposition 1.3. Let M be a smooth manifold with cohomology algebra $A := H^*(M; F)$. For any continuous map $f\colon M \to \mathbb{R}$ there exists a point $y \in \mathbb{R}$ such that the restriction homomorphism satisfies

$$\operatorname{rk}\left[H^k(M; F) \to H^k(f^{-1}(y); F)\right] \geq \max_{0 \leq m \leq |A^k|} \mathcal{N}_{A,k}(m).$$

Before we prove this proposition we need a simple

Lemma 1.4. Let M^n be a closed smooth manifold and $f\colon M \to \mathbb{R}$ a Morse function such that the critical points $p_1, \ldots, p_n$ have pairwise different critical values $f(p_i)$. For any $c \in \mathbb{R}$ let $M_c := f^{-1}(-\infty, c]$ and to any $C \subseteq M$ we assign the measure $\|C\|_k \in \mathbb{N}_0$ by

$$\|C\|_k := \operatorname{rk}\left[H^k(M; F) \to H^k(C; F)\right].$$

The function

$$\begin{aligned} \|M_-\|_k \colon \mathbb{R} &\to \mathbb{N}_0 \\ c &\mapsto \|M_c\|_k = \operatorname{rk}\left[H^k(M; F) \to H^k(f^{-1}(-\infty, c]; F)\right] \end{aligned}$$

assumes every value $0 \leq m \leq |H^k(M; F)|$.

Proof. Since the homotopy type of M_c changes only when c passes a critical value it is sufficient to prove that for any critical value c of f we have

$$\|M_{c-\varepsilon}\|_k \leq \|M_{c+\varepsilon}\|_k \leq \|M_{c-\varepsilon}\|_k + 1.$$

The first inequality holds because the $\|\cdot\|_k$-measure is monotonous. By Morse theory the pair $(M_{c-\varepsilon}, M_{c+\varepsilon})$ is homotopy equivalent to $(M_{c-\varepsilon} \cup e^l, M_{c-\varepsilon})$ where l is the index of the non-degenerate critical point $p_i \in M$ satisfying $f(p_i) = c$ and therefore

$$\dim H^k(M_{c-\varepsilon}, M_{c+\varepsilon}; F) \leq 1.$$

The long exact sequence of the pair $(M_{c+\varepsilon}, M_{c-\varepsilon})$ contains

$$\cdots \to H^k(M_{c-\varepsilon}, M_{c+\varepsilon}; F) \to H^k(M_{c+\varepsilon}; F) \to H^k(M_{c-\varepsilon}; F) \to \dots$$

which implies that the kernel of the restriction homomorphism $H^k(M_{c+\varepsilon}; F) \to H^k(M_{c-\varepsilon}; F)$ has dimension at most 1. The commutative diagram

$$\begin{array}{ccc} & & H^k(M_{y+\varepsilon}; F) \\ & \nearrow & \downarrow \\ H^k(M; F) & & \\ & \searrow & \\ & & H^k(M_{c-\varepsilon}; F) \end{array}$$

of restriction homomorphisms shows that $\|M_{c+\varepsilon}\|_k \leq \|M_{c-\varepsilon}\|_k + 1$. □

Proof of Proposition 1.3. We proceed by contradiction and assume that there exists a map $f\colon M \to \mathbb{R}$ with

$$\mathrm{width}_k(f) < \max_{0 \leq m \leq |A^k|} \mathcal{N}_{A,k}(m).$$

The continuous map f can be uniformly approximated by Morse functions f_n that fulfil the assumption of the preceding lemma, i.e. all critical points have different critical values. By the upper semi-continuity of width_k there exists a $N \gg 0$ such that $\mathrm{width}_k(f_N) \leq \mathrm{width}_k(f)$ so without loss of generality we can assume that f itself is a Morse function like in the Lemma above.

By Corollary 2.1.3 we can write

$$\begin{aligned} \mathrm{rk}\left[H^k(M; F) \to H^k(f^{-1}(y); F)\right] &= \left| A^k \big/ f_*\mu_M(\mathbb{R} \setminus \{y\}) \cap A^k \right| \\ &= \left| A^k \big/ \mu(\mathbb{R} \setminus \{y\}) \cap A^k \right| \end{aligned}$$

where $f_*\mu_M =: \mu$ is the pushforward of the standard ideal valued measure. By additivity we get

$$\mathrm{rk}\left[H^k(M; F) \to H^k(f^{-1}(y); F)\right] = \left| A^k \big/ (\mu(-\infty, y) + \mu(y, \infty)) \cap A^k \right|.$$

Multiplicativity implies that $\mu(-\infty, y) \cdot \mu(y, \infty) = 0$ hence $\mu(y, \infty) \subseteq \mu(-\infty, y)^\perp$. This implies

$$\begin{aligned} \mathrm{rk}\left[H^k(M; F) \to H^k(f^{-1}(y); F)\right] &\geq \left| A^k \big/ (\mu(-\infty, y) + \mu(-\infty, y)^\perp) \cap A^k \right| \\ &= \left| \partial^k \left(\mu(-\infty, y) \cap A^k\right) \right| \geq \mathcal{N}_{A,k}\left(\left|\mu(-\infty, y) \cap A^k\right|\right). \end{aligned}$$

Since f fulfils the assumptions of the preceding lemma the numbers $|\mu(-\infty, y)| = \|f^{-1}(-\infty, y]\|_k$

assumes every value $0 \leq m \leq |A^k|$ as $y \in \mathbb{R}$ varies. The claim follows by choosing y in such a way that $m_0 := \|f^{-1}(-\infty, y]\|_k$ has the property that $\mathcal{N}_{A,k}(m_0) = \max_{0 \leq m \leq |A^k|} \mathcal{N}_{A,k}(m)$. $\square$

Remark 1.5. Recall from Remark 1.4.2 that isoperimetric estimates yield waist inequalities in codimension 1, e.g. the waist of the sphere inequality. The proof of Proposition 1.3 resembles this philosophy. To the open subset $(-\infty, y)$ we assign the volume $\left|\mu(-\infty, y) \cap A^k\right|$ and to its boundary $\partial f^{-1}(-\infty, y) = f^{-1}(y)$ the volume $\left|A^k \Big/ \mu(\mathbb{R} \setminus \{y\}) \cap A^k\right|$. The proof above shows

$$\left|A^k \Big/ \mu(\mathbb{R} \setminus \{y\}) \cap A^k\right| \geq \mathcal{N}_{A,k}\left(\left|\mu(-\infty, y) \cap A^k\right|\right)$$

justifying why the function $\mathcal{N}_{A,k}$ from Definition 1.1 is called an isoperimetric profile.

2 Products of projective spaces

In this section the coefficient field is $\mathbb{Z}_2$. Let $M := \mathbb{RP}^{k_n} \times \cdots \times \mathbb{RP}^{k_1}$ be a product of projective spaces and

$$A := H^*(M; \mathbb{Z}_2) \cong \bigwedge[x_1, \ldots, x_n] \Big/ \left(x_1^{k_n+1}, \ldots, x_n^{k_1+1}\right)$$

its cohomology algebra. Each generator has degree 1 and let $K := \sum_i k_i$ denote the dimension of this product. This data shall be fixed for the rest of this subsection.

Using the obvious multi-index notation any $p \in A^k$ is given as a linear combination $p = \sum_a \lambda_a x^a$ where a is a k-element multisubset of the multiset $\mathbf{n} := \{1, \ldots, n\}$ where each element $1 \leq i \leq n$ has multiplicity $\mu(i) = k_i$. The set of such k-element multisubsets $a \subseteq \mathbf{n}$ is denoted by $\binom{\mathbf{n}}{k}$ and its cardinality by $\binom{k_n, \ldots, k_1}{k}$, in particular we have

$$\dim A^k = \binom{k_n, \ldots, k_1}{k}.$$

Remark 2.1. The coefficients $\binom{k_1, \ldots, k_n}{k}$ vanish iff the inequality $0 \leq k \leq K$ is violated, are symmetric in the variables k_i, satisfy the recursion

$$\binom{k_n, \ldots, k_1}{k} = \sum_{j=0}^{k_n} \binom{k_{n-1}, \ldots, k_1}{k-j}$$

and the symmetry

$$\binom{k_n, \ldots, k_1}{k} = \binom{k_n, \ldots, k_1}{K-k}.$$

Endow the set $\binom{\mathbf{n}}{k}$ with the *lexicographic total order*, i.e. two multisubsets $a, b \in \binom{\mathbf{n}}{k}$ satisfy $a < b$ iff their multiplicities satisfy $\mu_a(i) = \mu_b(i)$ for $i = n, n-1, \ldots, n-j+1$ and $\mu_a(n-j) < \mu_b(n-j)$ for some $0 \leq j < n$.

Definition 2.2. The *lexicographical bottom half* of $\binom{k_n,\dots,k_1}{k}$ is defined as the union of the following disjoint sets

$$\begin{aligned}
\mathcal{B}_n &:= \left\{ a \in \binom{k_n,\dots,k_1}{k} \,\middle|\, \mu_a(n) > \frac{k_n}{2} \right\} \\
\mathcal{B}_{n-1} &:= \left\{ a \in \binom{k_n,\dots,k_1}{k} \,\middle|\, \mu_a(n) = \frac{k_n}{2}, \mu_a(n-1) > \frac{k_{n-1}}{2} \right\} \\
&\ \vdots \\
\mathcal{B}_1 &:= \left\{ a \in \binom{k_n,\dots,k_1}{k} \,\middle|\, \mu_a(i) = \frac{k_i}{2} \text{ for each } i > 1 \text{ and } \mu_a(1) > \frac{k_1}{2} \right\}
\end{aligned}$$

The cardinality of this union is denoted by

$$\mathcal{H}\binom{k_n,\dots,k_1}{k} := |\mathcal{B}_n \,\dot\cup \dots \dot\cup\, \mathcal{B}_1|$$

Remark 2.3. (a) Some of the sets $\mathcal{B}_i$ can be empty, e.g. when k_i is odd then $\mathcal{B}_{i-1} = \mathcal{B}_{i-2} = \dots = \mathcal{B}_1 = \emptyset$.

(b) If $k_n = 2\beta - 1$ then

$$\mathcal{H}\binom{k_n,\dots,k_1}{k} = \left| \left\{ a \in \binom{\mathbf{n}}{k} \,\middle|\, \mu_a(n) \geq \beta \right\} \right| = \binom{\beta - 1, k_{n-1},\dots,k_1}{k - \beta}$$

(c) If $k_n = \dots = k_1 = 1$ then

$$\mathcal{H}\binom{k_n,\dots,k_1}{k} = \binom{k_{n-1},\dots,k_1}{k-1} = \binom{n-1}{k-1}$$

where the last symbol denotes an ordinary binomial coefficient.

The main result in this chapter is

Theorem 2.4. Let $k_n \leq \dots \leq k_1$, $k < \frac{K}{2}$ and $M := \mathbb{RP}^{k_n} \times \dots \times \mathbb{RP}^{k_1}$. Any continuous map $f\colon M \to \mathbb{R}$ admits a point $y \in \mathbb{R}$ such that the rank of the restriction homomorphism to the fiber $f^{-1}(y)$ satisfies

$$\operatorname{rk}\left[H^k(M;F) \to H^k(f^{-1}(y);F)\right] \geq \binom{k_n,\dots,k_1}{k} - 2\mathcal{H}\binom{k_n,\dots,k_1}{k}.$$

For every $p \in A^k \setminus \{0\}$ with $p = \sum_a \lambda_a x^a$ define

$$[p]_{\max} := \max\{a \,|\, \lambda_a \neq 0\} \in \binom{\mathbf{n}}{k}$$

where the maximum is taken with respect to the lexicographical order.

Lemma 2.5. If $p = \sum_a \lambda_a x^a, q = \sum_b \mu_b x^b \in A^k \setminus \{0\}$ satisfy $p \cdot q = 0$ then $a_0 := [p]_{max}$ and $b := [q]_{\max}$ satisfy $a_0 \cap b_0 \neq \emptyset$. Two multisubsets $a, b \subseteq \binom{\mathbf{n}}{k}$ are defined to have nonempty intersection iff for any $1 \leq i \leq n$ the multiplicities satisfy $\mu_a(i) + \mu_b(i) > k_i$.

Proof. Assume $[p]_{\max} \cap [q]_{\max} = \emptyset$, i.e. $\mu_{a_0}(i) + \mu_{b_0}(i) \leq k_i$. Define $a_0 \dot\cup b_0$ to be the multisubset of $\mathbf{n}$ such that the element $1 \leq i \leq n$ has multiplicity $\mu_{a_0}(i) + \mu_{b_0}(i)$. We claim that the coefficient of $x^{a_0 \dot\cup b_0}$ in the monomial expansion of the product $p \cdot q$ consists only of $\lambda_{a_0} \cdot \mu_{b_0}$. Assume there is another nonzero summand $\lambda_a \mu_b$ contributing to this coefficient, i.e. there are $a, b \in \binom{\mathbf{n}}{k}$ satisfying $a < a_0$ and $b < b_0$ and $a \cap b = \emptyset$ (otherwise the monomial product $x^a x^b$ already vanishes).

Sublemma 2.6. $a < a_0$ and $b < b_0$ imply $a \dot\cup b < a_0 \dot\cup b_0$.

Proof. Choose primes $p_1, \dots, p_n$ such that $p_i \geq p_{i+1}^{k_i+1} \cdot \dots \cdot p_n^{k_n+1}$ for every $1 \leq i < n$ and assign to each subset $l \in \binom{\mathbf{n}}{k}$ the norm $|L| := \prod_{i=1}^n p_i^{\mu_L(i)}$. This norm assigns different values to different subsets, is monotonic with respect to the lexicographic order and multiplicative with respect to union of multisubsets. $\square$

The conclusion of the preceding sublemma contradicts the assumption that the summand $\lambda_a \mu_b$ contributes to the coefficient of the monomial $x^{a_0 \dot\cup b_0}$. Hence in the product pq the monomial $x^{a_0 \dot\cup b_0}$ appears exactly with the coefficient $\lambda_{a_0} \mu_{b_0} \neq 0$ contradicting the assumption $pq = 0$. Therefore the very first assumption $a_0 \cap b_0 = \emptyset$ was incorrect yielding

$$[p]_{\max} \cap [q]_{\max} \neq \emptyset. \quad \square$$

For any linear subspace $I \subseteq A^k$ define

$$[I]_{\max} := \{[p]_{\max} | 0 \neq p \in I\} \subseteq \binom{\mathbf{n}}{k}.$$

Lemma 2.7.

$$\dim I = |[I]_{\max}|$$

Proof. Let $|[I]_{\max}| =: m$, i.e. $[I]_{\max} = \{a_1, \dots, a_m\}$ for some $a_1 < \dots < a_m$ elements of $\binom{\mathbf{n}}{k}$. For every a_i there exists an element $p_i \in I$ such that $[p_i]_{\max} = a_i$. We claim that the family (p_i) linearly spans I. For if $p \in I$ is an arbitrary we know that $[p]_{\max} \in [I]_{\max}$ hence $[p]_{\max} = a_{i_0}$ for some $1 \leq i_0 \leq m$. This shows that both p and p_{i_0} have the same leading monomial and there exists a scalar λ_{i_0} such that $p - \lambda_{i_0} p_{i_0} \in I$ and the multi-exponent of its leading monomial is strictly smaller than that of p. This procedure can be repeated showing that the (p_i) are indeed a generating set of I. Assume that there is a nontrivial linear relation of the form $\sum_i \lambda_i p_i = 0$. Equating the coefficients of the leading monomial of p_m yields first $\lambda_m = 0$ and then in turn consequently $\lambda_{m-1} = \dots = \lambda_1 = 0$. This finishes the proof of the claim that the (p_i) for a basis of I and hence Lemma 2.7. $\square$

Definition 2.8. Two families $\mathcal{A}, \mathcal{B} \subseteq \binom{\mathbf{n}}{k}$ are said to be *cross-intersecting* if any two $a \in \mathcal{A}, b \in \mathcal{B}$ have nonempty intersection.

Example 2.9. Lemma 2.5 shows that for any linear subspace $I \subseteq A^k$ the two families $\mathcal{A} := [I]_{\max} \subseteq \binom{\mathbf{n}}{k}$ and $\mathcal{B} := [I^\perp \cap A^k]_{\max} \subseteq \binom{\mathbf{n}}{k}$ are cross-intersecting.

Proposition 2.10. The separation profile of $A := \bigwedge[x_1, \dots, x_n]/(x_1^{k_1+1}, \dots, x_n^{k_n+1})$ satisfies

$$\mathcal{M}_{A,k}(m) = \max_{\substack{\mathcal{A},\mathcal{B} \subseteq \binom{\mathbf{n}}{k} \\ |\mathcal{A}|=m \\ \mathcal{A},\, \mathcal{B} \text{ cross-intersecting}}} |\mathcal{B}|$$

Proof. For any linear subspace $I \subseteq A^k$ of dimension m the families $\mathcal{A} := [I]_{\max}$ and $[I^\perp \cap A^k]_{\max}$ are subsets of $\binom{\mathbf{n}}{k}$, satisfy $|\mathcal{A}| = |[I]_{\max}| = \dim I = m$ and are cross-intersecting. Therefore the left hand side is smaller or equal than the right hand side.

On the other hand given two cross-intersecting families $\mathcal{A}, \mathcal{B} \subseteq \binom{\mathbf{n}}{k}$ such that $|\mathcal{A}| = m$ we can reverse the $[\cdot]_{\max}$ via $\langle\mathcal{A}\rangle := \langle x^J | J \in \mathcal{A}\rangle$ and similarly $\langle\mathcal{B}\rangle$. These linear subspaces of A^k satisfy $\dim\langle\mathcal{A}\rangle = |\mathcal{A}| = m$ and $\langle\mathcal{A}\rangle \cdot \langle\mathcal{B}\rangle = 0$ so $\mathcal{B} \subseteq \langle\mathcal{A}\rangle^\perp$ implying $\mathcal{M}_{A,k}(m) \geq |\mathcal{B}|$ which in turn implies that the left hand side ist greater or equal than the right hand side. $\square$

Example 2.11. The lexicographical bottom half $\mathcal{B} = \mathcal{B}_n \dot\cup \dots \dot\cup \mathcal{B}_1 \subseteq \binom{\mathbf{n}}{k}$ is cross-intersecting with itself.

It is tempting to ask the following

Question 2.12. Do any two cross-intersecting families $\mathcal{A}, \mathcal{B} \subseteq \binom{\mathbf{n}}{k}$ satisfy

$$|\mathcal{A}||\mathcal{B}| \leq \mathcal{H}\binom{k_n, \dots, k_1}{k}^2?$$

We only need the following special case of this inequality:

Proposition 2.13. Let $\mathcal{A}, \mathcal{B} \subseteq \binom{\mathbf{n}}{k}$ be cross-intersecting with $|A| = \mathcal{H}\binom{k_n,\dots,k_1}{k}$ then

$$|\mathcal{B}| \leq \mathcal{H}\binom{k_n, \dots, k_1}{k}.$$

Corollary 2.14.

$$\mathcal{M}_{A,k}\left(\mathcal{H}\binom{k_n, \dots, k_1}{k}\right) = \mathcal{H}\binom{k_n, \dots, k_1}{k}$$

Proof. Proposition 2.10 yields that the left hand side is less than or equal to the right hand side. The lexicographical bottom half establishes the equality. $\square$

In order to prove Proposition 2.13 we have to explain some machinery.

Definition 2.15. Let $\mathcal{A} \subseteq \binom{\mathbf{n}}{l}$ for some $1 \leq l \leq \sum_i k_i$ and let $1 \leq k \leq l$. The *k-shadow* $\Delta_k\mathcal{A}$ of $\mathcal{A}$ is defined as

$$\Delta_k\mathcal{A} := \left\{ B \,\middle|\, B \in \binom{\mathbf{n}}{k}, B \subseteq A \text{ for some } A \in \mathcal{A} \right\}.$$

Remark 2.16. Two families $\mathcal{A}, \mathcal{B} \subseteq \binom{\mathbf{n}}{k}$ are cross-intersecting iff

$$\mathcal{B} \subseteq \binom{\mathbf{n}}{k} \setminus \Delta_k(\mathcal{A}^c)$$

where $\mathcal{A}^c$ is the complementary family of $\mathcal{A}$, i.e.

$$\mathcal{A}^c := \{\mathbf{n} \setminus a | a \in \mathcal{A}\} \subseteq \binom{\mathbf{n}}{n-k}$$

and $\mathbf{n} \setminus a$ is defined via $\mu_{\mathbf{n}\setminus\mathbf{a}}(i) + \mu_a(i) = \mu_{\mathbf{n}}(i)$ for any $1 \leq i \leq n$.

The preceding observation yields that in order to give upper bounds for the sizes of cross-intersecting families as in Proposition 2.10 it suffices to give lower bounds on the size of the shadows $\Delta_k(\mathcal{A}^c)$. This is covered in the following generalisation of the Kruskal-Katona theorem.

Theorem 2.17. [Cle84] Let $1 \leq l \leq \sum k_i$. For $1 \leq \alpha \leq n$ define

$$\begin{bmatrix} \alpha \\ l \end{bmatrix} := \binom{k_\alpha, \ldots, k_1}{l}.$$

Any integer $1 \leq m \leq \begin{bmatrix} n \\ l \end{bmatrix} = |\binom{\mathbf{n}}{l}|$ has a unique representation in the form

$$m = \begin{bmatrix} \alpha(l) \\ l \end{bmatrix} + \begin{bmatrix} \alpha(l-1) \\ l-1 \end{bmatrix} + \cdots + \begin{bmatrix} \alpha(t) \\ t \end{bmatrix} \tag{2.1}$$

where $t > 0$, $\alpha(l) \geq \alpha(l-1) \geq \cdots \geq \alpha(t)$, $\begin{bmatrix} \alpha(t) \\ t \end{bmatrix} > 0$ and whenever $t < i \leq l$ and e satisfy $\alpha(i) = \alpha(i-1) = \cdots = \alpha(i-e)$ then $e < k_{\alpha(i)+1}$. We will refer to (2.1) as the l-expansion of m.

Further, if $\mathcal{A} \subseteq \binom{\mathbf{n}}{l}$ consists of m subsets, then

$$|\Delta_k \mathcal{A}| \geq \begin{bmatrix} \alpha(l) \\ k \end{bmatrix} + \cdots + \begin{bmatrix} \alpha(t) \\ k-l+t \end{bmatrix}.$$

The following calculations are essentially from [And88]. As we could not directly cite any result from there and the paper contains some typing errors we proceed to carry out the calculations again.

Proof of Proposition 2.13. Let $k_n = 2\alpha_n$, $k_{n-1} = 2\alpha_{n-1}$, ..., $k_{n-r+1} = 2\alpha_{n-r+1}$ and $k_{n-r} = 2\beta - 1$. With the terminology from Definition 2.2 we can conclude $\mathcal{B}_{n-r-1} = \cdots = \mathcal{B}_1 = \emptyset$

and therefore

$$
\begin{aligned}
&\mathcal{H}\binom{k_n, \dots, k_1}{k} = |\mathcal{B}_n| + \dots + |\mathcal{B}_{n-r}| \\
=&\binom{\alpha_n - 1, k_{n-1}, \dots, k_1}{k - \alpha_n - 1} + \dots + \binom{\alpha_{n-r+1} - 1, 2\beta - 1, \dots, k_1}{k - \alpha_n - \dots - \alpha_{n-r+1} - 1} + \binom{\beta - 1, k_{n-r-1}, \dots, k_1}{k - \alpha_n - \dots - \alpha_{n-r+1} - \beta} \\
=&\binom{\alpha_n - 1, k_{n-1}, \dots, k_1}{K - k} + \dots + \binom{\alpha_{n-r+1} - 1, 2\beta - 1, \dots, k_1}{K - k - \alpha_n - \dots - \alpha_{n-r+2}} + \binom{\beta - 1, k_{n-r-1}, \dots, k_1}{K - k - \alpha_n - \dots - \alpha_{n-r+1}}
\end{aligned}
$$

by the symmetry relation noted in Remark 2.1. The recursion identity from the very same remark yields

$$
\begin{aligned}
&\mathcal{H}\binom{k_n, \dots, k_1}{k} \\
=&\sum_{i=0}^{\alpha_n - 1} \binom{k_{n-1}, \dots, k_1}{K - k - i} + \dots + \sum_{i=0}^{\alpha_{n-r+1} - 1} \binom{2\beta - 1, \dots, k_1}{K - k - \sum\limits_{j=n-r+2}^{n} \alpha_j - i} + \sum_{i=0}^{\beta - 1} \binom{k_{n-r-1}, \dots, k_1}{K - k - \sum\limits_{j=n-r+1}^{n} \alpha_j - i} \\
=&\begin{bmatrix} n-1 \\ K-k \end{bmatrix} + \dots + \begin{bmatrix} n-1 \\ K-k-(\alpha_n - 1) \end{bmatrix} + \dots + \begin{bmatrix} n-r \\ K-k-\sum_{j=n-r+2}^{n} \alpha_n \end{bmatrix} + \dots \\
&+ \begin{bmatrix} n-r \\ K-k-\sum_{j=n-r+1}^{n} +1 \end{bmatrix} + \begin{bmatrix} n-r-1 \\ K-k-\sum_{j=n-r+1}^{n} \alpha_j \end{bmatrix} + \dots + \begin{bmatrix} n-r-1 \\ K-k-\sum_{j=n-r+1}^{n} -(\beta - 1) \end{bmatrix}
\end{aligned}
$$

This seems like the $(K-k)$-expansion of $\mathcal{H}\binom{k_n, \dots, k_1}{k} = |\mathcal{A}| = |\mathcal{A}^c|$. Indeed the choice

$$
\begin{aligned}
\alpha(K-k) = \dots = \alpha(K-k-(\alpha_n - 1)) &= n-1 \\
&\vdots \\
\alpha\left(K-k-\sum_{j=n-r+2}^{n} \alpha_j\right) = \dots = \alpha\left(K-k-\sum_{j=n-r+2}^{n} \alpha_j - (\alpha_{n-r+1} - 1)\right) &= n-r \\
\alpha\left(K-k-\sum_{j=n-r+1}^{n} \alpha_j\right) = \dots = \alpha\left(K-k-\sum_{j=n-r+1}^{n} \alpha_j - (\beta - 1)\right) &= n-r-1
\end{aligned}
$$

also satisfies the condition about the number of consecutive $\alpha(i)$ which are allowed to be equal since

$$
\begin{aligned}
\alpha_n - 1 &< k_n \\
&\vdots \\
\alpha_{n-r+1} - 1 &< k_{n-r+1} \\
\beta - 1 &< k_{n-r}.
\end{aligned}
$$

Now Theorem 2.17 implies

$$|\mathcal{A}^c| \geq \sum_{l=n-r+1}^{n} \underbrace{\sum_{i=0}^{\alpha_l - 1} \begin{bmatrix} l-1 \\ k - \sum_{j=l+1}^{n} \alpha_j - i \end{bmatrix}}_{=:T_l} + \underbrace{\sum_{i=0}^{\beta-1} \begin{bmatrix} n-r-1 \\ k - \sum_{j=n-r+1}^{n} \alpha_j - i \end{bmatrix}}_{=:T_{n-r}}. \tag{2.2}$$

Sublemma 2.18.

$$\mathcal{H}\binom{k_n, \dots, k_1}{k} + \sum_{l=n-r}^{n} T_l = \begin{bmatrix} n \\ k \end{bmatrix}$$

Proof. Set

$$\mathcal{H}\binom{k_n, \dots, k_1}{k} = \sum_{l=n-r+1}^{n} \underbrace{\sum_{i=0}^{\alpha_l - 1} \begin{bmatrix} l-1 \\ K - k - \sum_{j=l+1}^{n} \alpha_j - i \end{bmatrix}}_{=:S_l} + \underbrace{\sum_{i=0}^{\beta-1} \begin{bmatrix} n-r-1 \\ K - k - \sum_{j=n-r+1}^{n} \alpha_j - i \end{bmatrix}}_{=:S_{n-r}}.$$

We have

$$\begin{aligned} T_{n-r} + S_{n-r} &= \sum_{i=0}^{\beta-1} \begin{bmatrix} n-r-1 \\ k - \sum_{j=n-r+1}^{n} \alpha_j - i \end{bmatrix} + \sum_{i=0}^{\beta-1} \begin{bmatrix} n-r-1 \\ K - k - \sum_{j=n-r+1}^{n} \alpha_j - i \end{bmatrix} \\ &= \sum_{i=0}^{\beta-1} \begin{bmatrix} n-r-1 \\ k - \sum_{j=n-r+1}^{n} \alpha_j - i \end{bmatrix} + \sum_{i=0}^{\beta-1} \begin{bmatrix} n-r-1 \\ k - \sum_{j=n-r+1}^{n} \alpha_j - (2\beta - 1) + i \end{bmatrix} \text{ by symmetry} \\ &= \sum_{i=0}^{2\beta-1} \begin{bmatrix} n-r-1 \\ k - \sum_{j=n-r+1}^{n} \alpha_j - i \end{bmatrix} = \begin{bmatrix} n-r \\ k - \sum_{j=n-r+1}^{n} \alpha_j \end{bmatrix} \end{aligned}$$

as well as

$$\begin{aligned} T_{n-r+1} + S_{n-r+1} &= \sum_{i=0}^{\alpha_{n-r+1}} \begin{bmatrix} n-r \\ k - \sum_{j=n-r+2}^{n} \alpha_j - i \end{bmatrix} + \sum_{i=0}^{\alpha_{n-r+1}} \begin{bmatrix} n-r \\ K - k - \sum_{j=n-r+2}^{n} \alpha_j - i \end{bmatrix} \\ &= \sum_{i=0}^{\alpha_{n-r+1}} \begin{bmatrix} n-r \\ k - \sum_{j=n-r+2}^{n} \alpha_j - i \end{bmatrix} + \sum_{i=0}^{\alpha_{n-r+1}} \begin{bmatrix} n-r \\ k - \sum_{j=n+r+2}^{n} \alpha_j - 2\alpha_{n-r+1} + i \end{bmatrix} \text{ again by symmetry} \\ &= \sum_{\substack{i=0 \\ i \neq \alpha_{n-r+1}}}^{2\alpha_{n-r+1}} \begin{bmatrix} n-r \\ k - \sum_{j=n-r+2}^{n} \alpha_j - i \end{bmatrix} \end{aligned}$$

implying

$$T_{n-r+1} + S_{n-r+1} + T_{n-r} + S_{n-r} = \begin{bmatrix} n-r+1 \\ k - \sum_{j=n-r+2}^{n} \alpha_j \end{bmatrix}.$$

Repeating this prodecure we get

$$\sum_{l=n-r}^{n} S_l + T_j = \begin{bmatrix} n \\ k \end{bmatrix}$$

proving the sublemma. ☐

Back to the proof of Proposition 2.13. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting we have $\mathcal{B} \subseteq \binom{\mathbf{n}}{k} \setminus \Delta_k(\mathcal{A}^c)$ and hence by estimate (2.2) and the preceding sublemma

$$|\mathcal{B}| \leq \binom{\mathbf{n}}{k} - \sum_{l=n-r}^{n} T_l = \mathcal{H}\binom{k_n, \dots, k_1}{k}. \qquad \square$$

We conclude with the

Proof of Theorem 2.4. By Proposition 1.3 there exists a point $y \in \mathbb{R}$ such that

$$\operatorname{rk}\left[H^k(M;F) \to H^k(f^{-1}(y);F)\right] \geq \max_{0 \leq m \leq |A^k|} \mathcal{N}_{A,k}(m) \geq \mathcal{N}_{A,k}\left(\mathcal{H}\binom{k_n, \dots, k_1}{k}\right).$$

Using Remark 1.2 and Corollary 2.14 we get

$$\begin{aligned} \mathcal{N}_{A,k}\left(\mathcal{H}\binom{k_n, \dots, k_1}{k}\right) &\geq |A^k| - \mathcal{H}\binom{k_n, \dots, k_1}{k} - \mathcal{M}_{A,k}\left(\mathcal{H}\binom{k_n, \dots, k_1}{k}\right) \\ &= \binom{k_n, \dots, k_1}{k} - 2\mathcal{H}\binom{k_n, \dots, k_1}{k}. \quad \square \end{aligned}$$

Remark 2.19. (i) We only used the specific structure of the $H^*(\mathbb{RP}^{k_n} \times \ldots \times \mathbb{RP}^{k_1}; \mathbb{Z}_2)$ so Theorem 2.4 holds for every manifold M satisfying

$$H^*(M;\mathbb{Z}_2) \cong H^*(\mathbb{RP}^{k_n} \times \ldots \times \mathbb{RP}^{k_1}; \mathbb{Z}_2).$$

Examples of such manifolds which are not diffeomorphic to $\mathbb{RP}^{k_n} \times \ldots \times \mathbb{RP}^{k_1}$ can be constructed by replacing subproducts $S := \mathbb{RP}^{k_{i_r}} \times \ldots \times \mathbb{RP}^{k_{i_1}}$ by the connected sum $S \# \Sigma$ with a nontrivial $\mathbb{Z}_2$-homology sphere Σ.

(ii) Gromov also asks [Gro10, p. 509] whether Theorem 1.3.2 can be generalised to products of higher-dimensional spheres $M := S^{k_n} \times \ldots \times S^{k_1}$. Their cohomology satisfies

$$A := H^*(M;F) \cong \bigwedge[x_1, \dots, x_n] \Big/ (x_1^2, \dots, x_n^2)$$

where each generator x_i has degree k_i. The dimension of A^k is the number of subsets $a \subseteq \mathbf{n}$ of *weight* $|a| = k$. It is however not clear how to generalise the preceding combinatorics to such sets with weighted elements.

4 Higher codimensions

As already announced in Section 1.3 in this chapter we will prove cohomological waist inequalities in arbitrary codimensions $q \geq 1$ using a filling argument like in Section 1.4.

Theorem 2.4. Let N be a q-manifold. Every continuous map $f\colon T^n \to N$ admits a point $y \in N$ such that

$$\operatorname{rk}\left[H^1(T^n;\mathbb{Z}) \to H^1(f^{-1}(y);\mathbb{Z})\right] \geq n-q.$$

It is easy to construct maps $f\colon T^n \to N^q$ such that all fibers of f are the disjoint union of at most 2^q homotopic $(n-q)$-tori (cf. Remark 2.2 (ii)) making the theorem above the first cohomological waist inequality which is sharp. It also generalises to source manifolds that need not be tori but can be arbitrary essential m-manifolds with fundamental group $\mathbb{Z}^n$ (cf. Theorem 5.2). Using rational homotopy theory we could also prove the following estimate about cartesian powers of higher-dimensional spheres.

Theorem 6.1. Let $p \geq 3$ be odd and $n \leq p-2$. Consider $M = (S^p)^n$ or any simply connected, closed manifold of dimension pn with the rational homotopy type $(S^p)^n_{\mathbb{Q}}$ and an orientable manifold N^q. Every continuous map $f\colon M \to N$ admits a point $y \in N$ such that

$$\operatorname{rk}\left[H^p(M;\mathbb{Q}) \to H^p(f^{-1}(y);\mathbb{Q})\right] \geq n-q.$$

Theorem 6.1 is the first lower bound on width_p with $p > 1$ that has been proven using a filling argument.

1 Preliminaries

We will deal with various kinds of manifolds such as smooth manifolds, topological manifolds and manifolds with corners. If any specifier is missing by a manifold we mean a smooth manifold.

Definition 1.1 (Manifolds with corners)**.** Let C^n be a topological n-manifold with boundary and let

$$\overline{\mathbb{R}}^n_+ := [0,\infty)^n.$$

A pair (U,φ) is called a *chart with corners* for M iff φ is a homeomorphism $\varphi\colon U \to V$ from some open subset $U \subseteq C$ to some relatively open subset $V \subseteq \overline{\mathbb{R}}^n_+$. Two such charts with corners (U,φ) and (V,ψ) are called *smoothly compatible* iff the transition map

$$\varphi \circ \psi^{-1}\colon \psi(U \cap V) \to \varphi(U \cap V)$$

is a diffeomorphism.

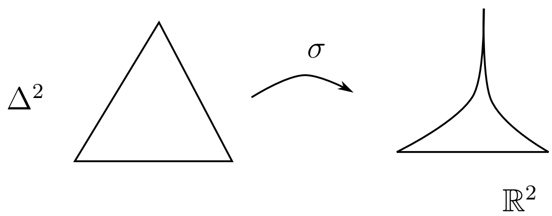
A *smooth structure with corners* on C is a maximal collection of smoothly compatible charts with corners which cover all of C. A *smooth manifold with corners* is a topological manifold C together with such a smooth structure with corners.

The *codimension k corner of* $\overline{\mathbb{R}}^n_+$ is the subset of all points $\overline{\mathbb{R}}^n_+$ where exactly k coordinates vanish. A point $p \in M$ is said to be a *codimension k corner point of C* if its image under a smooth chart with corners is a codimension k corner point of $\overline{\mathbb{R}}^n_+$. This notion does not depend on the choice of the chart [Lee03, Lemma 10.29]. The boundary of C is precisely the union of all corner points of positive codimension.

An example of a smooth manifold with corners up to codimension k is the standard k-simplex Δ^k. Another source of examples will given in the following proposition.

Definition 1.2 (Smooth, embedded simplices)**.** Let N^q be a manifold. A *smooth, embedded k-simplex σ in N* is a smooth map $\sigma\colon \Delta^k \to N$ such that there exists an open neighbourhood $\Delta^k \subset U \subset \mathbb{R}^k$ and a smooth extension $\widetilde{\sigma}\colon U \to N$ which is an embedding.

The figure below illustrates that for a smooth map $\sigma\colon \Delta^k \to N$ the condition above is stronger than merely being a topological embedding.



Definition 1.3 (Stratum transversality)**.** Let M^n and N^q be manifolds without boundary, $f\colon M \to N$ smooth and $\sigma\colon \Delta^k \to N$ a smooth, embedded simplex. We say that f intersects σ *stratum transversally* if σ and all of its faces intersect f transversally.

If f intersects σ stratum transversally the same holds for all faces of σ.

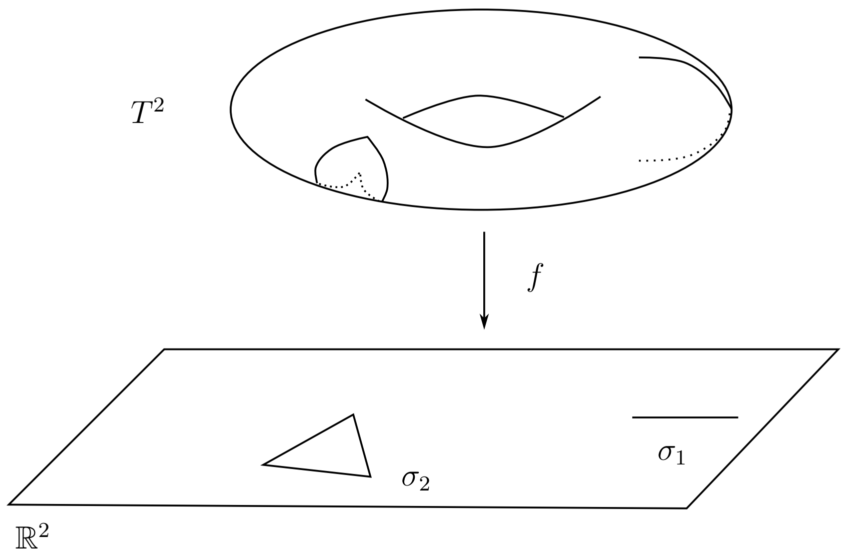
Proposition 1.4 (Generic preimages of simplices)**.** Let M^n and N^q be closed, oriented manifolds, $\sigma\colon \Delta^k \to N$ a smooth, embedded simplex and $f\colon M \to N$ a smooth map intersecting σ stratum transversally.

The preimage $f^{-1}\sigma(\Delta^k)$ is an oriented topological $(n-q+k)$-manifold with boundary

$$\partial f^{-1}\sigma(\Delta^k) = f^{-1}\sigma(\partial\Delta^k).$$

Proof. Theorem 3 in [Nie82] shows that $f^{-1}\sigma(\Delta^k)$ is a smooth manifold with corners up to codimension k hence it is a topological manifold with boundary. Note that most of the technical assumptions are met since M and N do not have boundary. Moreover the theorem states that the codimension l corner points of $f^{-1}\sigma(\Delta^k)$ are precisely the preimages of codimension l corner points of Δ^k, in particular $\partial f^{-1}\sigma(\Delta^k) = f^{-1}\sigma(\partial\Delta^k)$. □

Example 1.5. Let $f\colon T^2 \to \mathbb{R}^2$ be the projection depicted in the figure below. It shows the preimages of the 1-simplex σ_1 and the 2-simplex σ_2. Note that the preimages of some faces can be empty.



Mind the following notational convention.

Notation 1.6. In the situation of Proposition 1.4 we frequently denote the preimage of a simplex $\sigma \in \mathcal{T}_k$ by

$$F_\sigma := f^{-1}\sigma(\Delta^k)$$

and similarly

$$F_{\partial\sigma} := f^{-1}\sigma(\partial\Delta^k) = \partial F_\sigma.$$

We will often use this notation without explicitly mentioning it.

Definition 1.7 (Smooth triangulations)**.** Let N^q be a smooth manifold. A *smooth triangulation* $\mathcal{T} = (K, \varphi)$ *of* N consists of a finite simplicial complex K together with a homeomorphism $\varphi\colon |K| \to N$ such that the restriction of φ to any simplex yields a smooth, embedded simplex in N. The set of all of these smooth k-simplices of $\mathcal{T}$ shall be denoted by $\mathcal{T}_k$. We will often omit the specification *smooth* and simply talk about *a triangulation* and its *simplices.*

Let R be a coeffcient ring. If N^q is R-oriented a triangulation $\mathcal{T}$ is called R*-oriented* iff the sum of the elements in $\mathcal{T}_q$, i.e. the top-dimensional simplices, represents the R-oriented fundamental class of N^q.

Any smooth manifold N admits a smooth triangulation [Mun67, Theorem 10.6].

Proposition 1.8. Let $f\colon M^n \to N^q$ be a smooth map between closed R-oriented manifolds, $\mathcal{T}$ an R-oriented triangulation of N such that f intersects all the simplices $\sigma \in \mathcal{T}_q$ stratum transversally. For $k = 0, \dots, q$ we can inductively assign singular chains $c_\sigma \in C_{n-q+k}(F_\sigma; R)$ to every $\sigma \in \mathcal{T}_k$ such that the following properties hold.

(i) For $\sigma \in \mathcal{T}_0$ the chain $c_\sigma \in C_{n-q}(F_\sigma; R)$ represents the (correctly oriented) fundamental class of F_σ.

(ii) For $1 \leq k \leq q$ and $\sigma \in \mathcal{T}_k$ we can view the sum

$$\sum_{i=0}^{k} (-1)^i c_{\partial_i \sigma} \tag{1.1}$$

as an element of $C_{n-q+k-1}(\partial F_\sigma; R)$ and this represents the (correctly oriented) fundamental class of ∂F_σ with the boundary orientation. The element $c_\sigma \in C_{n-q+k}(F_\sigma; R)$ satisfies

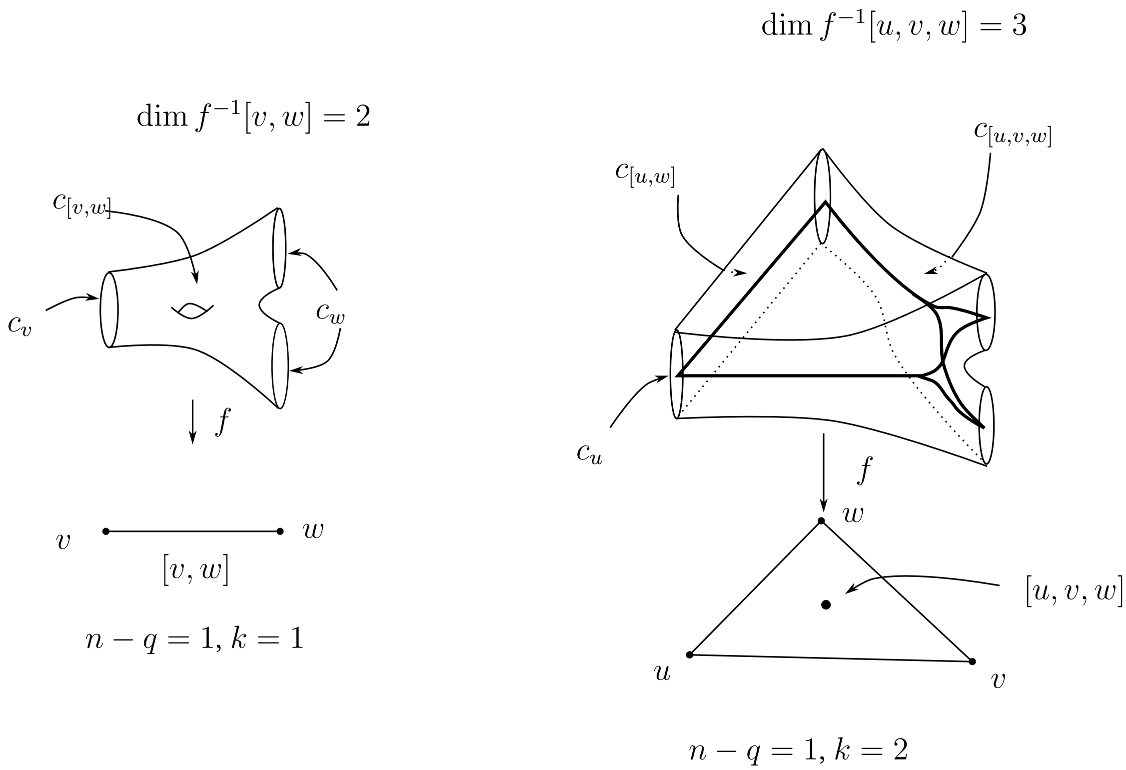
$$\partial c_\sigma = \sum_{i=0}^{k} (-1)^i c_{\partial_i \sigma} \tag{1.2}$$

as an equation in $C_{n-q+k-1}(F_\sigma; R)$ and c_σ represents the (correctly oriented) relative fundamental class in $H_{n-q+k}\left(F_\sigma, \partial F_\sigma; R\right)$.

(iii) The sum

$$\sum_{\sigma \in \mathcal{T}_q} c_\sigma \in C_n(M; R) \tag{1.3}$$

represents the (correctly oriented) fundamental class of M.



In the example picture on the right hand side $c_{[u,w]}$ is a cylinder and both $c_{[u,v]}$ and $c_{[v,w]}$ are pairs of pants. The chain $c_{[u,v,w]}$ is a solid double torus. The bold line is mapped to the barycentre of $[u, v, w]$ and the farther a point in $c_{[u,v,w]}$ is from this core line the closer it is mapped to $\partial[u, v, w]$.

Remark 1.9. Technically the summands appearing in the expressions (1.1), (1.2) and (1.3) are elements of different chain groups $C_{n-q+k-1}F_{\partial_i\sigma}$ (for varying i) or C_nF_σ (for varying σ). In order to make sense of the sums and equations we view these summands as chains in the chain group of the larger space $F_{\partial\sigma}$ or M. For the sake of legibility we omit all the inclusions and their induced maps on chain groups and ask the reader to interpret such equations of cycles in a sensible way. This convention holds for the rest of this paper.

Proof of Proposition 1.8. Proposition 1.4 shows that for all $\sigma \in \mathcal{T}_k$ the preimage F_σ is an oriented topological $(n-q+k)$-manifold with boundary $F_{\partial\sigma}$. Hence the notion of fundamental classes makes sense. Bear in mind that both F_σ and ∂F_σ may be empty or have several components.

(i) For every $\sigma \in \mathcal{T}_0$ the preimage F_σ is a closed oriented $(n-q)$-dimensional submanifold of M and it is easy to arrange (i). We proceed by induction over k and assume that we have constructed chains c_τ for all simplices $\tau \in \mathcal{T}_l$ of dimension $l < k$.

(ii) A standard calculation shows

$$\partial \sum_{i=0}^{k}(-1)^i c_{\partial_i\sigma} = \sum_{i=0}^{k}(-1)^i \partial c_{\partial_i\sigma} = \sum_{i=0}^{k}(-1)^i \sum_{j=0}^{k-1}(-1)^j c_{\partial_j\partial_i\sigma} = 0.$$

Hence $\sum_{i=0}^{k}(-1)^i c_{\partial_i\sigma}$ defines a cohomology class in $H_{n-q+k-1}(\partial F_\sigma)$. For every $0 \leq j \leq k$ the induced maps of the inclusions satisfy

$$H_{n-q+k-1}(\partial F_\sigma) \to H_{n-q+k-1}\left(\partial F_\sigma, \bigcup_{i\neq j} F_{\partial_i\sigma}\right)$$
$$\left[\sum_{i=0}^{k}(-1)^i c_{\partial_i\sigma}\right] \mapsto \left[(-1)^j c_{\partial_j\sigma}\right].$$

For every $p \in F_{\partial_j\sigma}$ the image of these classes in $H_{n-q+k-1}(F_\sigma, F_\sigma \setminus p)$ is the correct local orientation of $F_{\partial_j\sigma}$ in the point p where $F_{\partial_j\sigma} \subseteq \partial F_\sigma$ is oriented as the boundary of F_σ. This proves that $\sum_{i=0}^{k}(-1)^i c_{\partial_i\sigma}$ represents the (correctly oriented) fundamental class of $F_{\partial\sigma}$.

The fundamental class $[c_\sigma] \in H_{n-q+k}(F_\sigma, \partial F_\sigma)$ satisfies

$$\partial\colon H_{n-q+k}(F_\sigma, \partial F_\sigma) \to H_{n-q+k-1}(\partial F_\sigma) \tag{1.4}$$
$$[c_\sigma] \mapsto \left[\sum_{i=0}^{k}(-1)^i c_{\partial_i\sigma}\right] \tag{1.5}$$

and the relative cycle c_σ can be modified so as to achieve equation (1.2) on chain level.

(iii) We have

$$\partial \sum_{\sigma\in\mathcal{T}_q} c_\sigma = \sum_{\sigma\in\mathcal{T}_q}\sum_{i=0}^{q}(-1)^i c_{\partial_i\sigma} = 0$$

since every $(q-1)$-simplex is the face of exactly two q-simplices and inherits different orientations from them. Hence $\sum_{\sigma\in\mathcal{T}_q} c_\sigma$ defines a homology class in $H_n M$. Again for every $\tau \in \mathcal{T}_q$ the inclusion $(M, \emptyset) \to \left(M, \bigcup_{\sigma\in\mathcal{T}\setminus\tau} F_\sigma\right)$ satisfies

$$H_n(M) \to H_n\left(M, \bigcup_{\sigma\in\mathcal{T}\setminus\tau} F_\sigma\right)$$
$$\left[\sum_{\sigma\in\mathcal{T}_q} c_\sigma\right] \mapsto c_\tau$$

and for every $p \in F_\tau$ arbitrary the image of these classes in $H_n(M, M\setminus p)$ yields the correct local orientation of M in p. □

The rest of this section is devoted to the formulation and proof of Proposition 1.11, a genericity result which for any map $f\colon M \to N$ guarantees the existence of a triangulation of the target manifold N which is (in a precise sense) generic and fine.

Lemma 1.10. Let M and N be manifolds without boundary. We will denote the space of all continuous maps $f\colon M \to N$ by $C^0(M,N)$ and it shall be equipped with the compact-open topology. If M is compact the subspace topology on $C^\infty(M,N) \subset C^0(M,N)$ is coarser than the weak C^∞-topology.

Proof. We will recall the weak C^∞-topology by describing a subbasis. Let $1 \le r < \infty$, $f \in C^\infty(M,N)$, (φ, U), (ψ, V) charts on M and N; let $K \subset U$ be compact such that $f(K) \subset V$ and let $0 < \varepsilon \le \infty$. Define

$$\mathcal{N}^r(f; (\varphi, U), (\psi, V), K, \varepsilon)$$

to be the set of all smooth maps $g\colon M \to N$ such that $g(K) \subset V$ and

$$\|D^k(\psi g \varphi^{-1})(x) - D^k(\psi f \varphi^{-1})(x)\| < \varepsilon$$

for all $x \in \varphi(K)$ and $|k| \leq r$.

Let $K \subseteq M$ be compact and $W \subseteq N$ open. We have to show that the set

$$\mathcal{U}(K,U) := \{g \in C^\infty(M,N) | g(K) \subseteq U\}$$

is open in the weak C^∞-topology. Let $f \in \mathcal{U}(K,U)$. Choose finitely many charts (φ_i, U_i) of M together with compact subsets $K_i \subset U_i$ such that $f(U_i)$ lies in the domain of some chart (ψ_i, V_i) of N (indexed over the same set) and $\bigcup_i K_i = M$. We have

$$f \in \bigcap_i \mathcal{N}^1(f; (\varphi_i, U_i), (\psi_i | V_i \cap W, V_i \cap W), K_i \cap K, \infty) \subseteq \mathcal{U}(K,U)$$

proving the claim. □

Proposition 1.11. Let M and N be two closed manifolds. For every smooth map $f\colon M \to N$ and every open cover $\mathcal{U} = (U_i)_{i \in I}$ of N there exists a smooth triangulation $\mathcal{T}$ of N and a sequence of smooth maps $f_n\colon M \to N$ uniformly converging to f such that the following properties hold:

(i) Every map f_n intersects every simplex $\sigma \in \mathcal{T}_k$ stratum transversally.

(ii) For every $\sigma \in \mathcal{T}_k$ there exists an index $i \in I$ such that $\sigma(\Delta^k) \subseteq U_i$.

Proof. Choose a smooth triangulation $\mathcal{T} = (K, \varphi)$ of N and consider the preimage $\varphi^{-1}\mathcal{U} := (\varphi^{-1}U_i)_{i \in I}$ which is an open cover of $|K|$. Since N is compact this open cover has a Lebesgue number with respect to some standard metric on $|K|$. After barycentric subdivision we can assume that every simplex of $|K|$ is contained in some $f^{-1}U_i$, i.e. its image is contained in U_i.

For every smooth, embedded simplex $\sigma\colon \Delta^k \to N$ the subset

$$\{f \in C^\infty(M,N) | f \pitchfork \operatorname{im} \sigma\} \subseteq C^\infty(M,N)$$

is a residual in the weak C^∞-topology, i.e. it is the countable intersection of open and dense subsets [Hir76, Transversality Theorem 2.1]. Moreover the Baire category theorem applies to the weak C^∞-topology, i.e. every residual set is dense. The set

$$\begin{aligned} &\{g \in C^\infty(M,N) | \text{every simplex intersects } g \text{ stratum transversally}\} \\ = &\bigcap_{\sigma \text{ simplex of } \mathcal{T}} \{g \in C^\infty(M,N) | \sigma \text{ intersects } g \text{ stratum transversally}\} \end{aligned}$$

is the countable intersection of residual sets, hence itself residual and therefore dense. Since the compact-open topology is coarser than the weak C^∞-topology the claim follows. □

2 The main inequality

For the rest of this chapter N^q always denotes a smooth q-manifold. At the beginning we allow N to be disconnected, to have non-empty boundary or to be non-compact. Theorem 2.1 will hold for this general class of target manifolds. But we will quickly see that we can restrict ourselves to the case where N is closed and connected.

By cohomology we mean Čech cohomology and it will be denoted by H^* (and not by $\check{H}^*$ like in the last chapter). Without further notice we will use the comparison theorem 1.6 stating that there is a natural isomorphism between Čech and ordinary (i.e. singular or cellular) cohomology on the category of CW pairs. At the beginning the coefficient ring is $R = \mathbb{Z}$. Later on we will restrict ourselves to $R = \mathbb{Q}$ but this will be indicated. In this thesis T^n always denotes the n-dimensional torus.

Theorem 2.1. Every continuous map $f\colon T^n \to N^q$ admits a point $y \in N^q$ such that the rank of the restriction homomorphism satisfies

$$\operatorname{rk}\left[H^1(T^n;\mathbb{Z}) \to H^1(f^{-1}(y);\mathbb{Z})\right] \geq n - q.$$

Remark 2.2. (i) This inequality is non-vacuous only if $n > q$ which we will tacitly assume from now on. Furthermore it shows $\operatorname{width}_1(T^n/N) \geq n - q$.

(ii) Choose a projection $T^n \to T^q$, a continuous function $a\colon S^1 \to \mathbb{R}$ such that every point has at most two preimages and and an embedding $\mathbb{R}^q \hookrightarrow N$. Consider the composition

$$f\colon T^n \to T^q \xrightarrow{a^q} \mathbb{R}^q \hookrightarrow N.$$

Every fiber of this map is the disjoint union of at most 2^q homotopic $(n-q)$-tori. This map proves that the inequality above is sharp, independent of the target manifold N. In particular we get $\operatorname{width}_1(T^n/N) = n - q$.

(iii) Let us assume for the moment that we have proven the theorem for closed connected N. We will explain how the theorem extends to manifolds which are possibly disconnected, non-compact or have non-empty boundary. Since T^n is connected we can restrict the target of f to the component which is hit. If N had boundary consider the inclusion $N \hookrightarrow D$ into the double D of N. Since D has no boundary we can apply the theorem to the composition

$$T^n \xrightarrow{f} N \hookrightarrow D$$

yielding the theorem for N.

If N is non-compact we choose a sequence $N_1 \subset N_2 \subset \ldots \subset N$ such that each N_i is a smooth compact codimension 0 submanifold with boundary and $\bigcup_{i=1}^{\infty} \operatorname{int} N_i = N$ (such an exhaustion exists by a strong form of the Whitney embedding theorem where every (even non-compact) manifold can be embedded into some $\mathbb{R}^N$ with closed image). Since $f(T^n)$ is compact it is contained in N_i for some $i \gg 0$, i.e. we can view f as a map $T^n \to N_i$ and we already deduced the theorem for compact manifolds with boundary. For the rest of this paper we will assume the target manifold N to be closed and connected.

The theorem will essentially follow from the following

Proposition 2.3. Let $f\colon T^n \to N^q$ be a smooth map where N is a closed manifold together with a smooth triangulation $\mathcal{T}$ the simplices of which intersect f stratum transversally. Then there exists a simplex $\sigma \in \mathcal{T}_k$ such that the preimage $F_\sigma := f^{-1}\sigma(\Delta^k)$ satisfies

$$\operatorname{rk}\left[H^1(T^n;\mathbb{Z}) \to H^1(F_\sigma;\mathbb{Z})\right] \geq n-q.$$

Proof of Theorem 2.1 assuming Proposition 2.3. Assume there is a continuous map $f\colon T^n \to N^q$ such that $\operatorname{width}_1(f) < n-q$. Since T^n is compact the standard ideal valued measure μ_{T^n} on T^n satisfies the continuity axiom. The same holds for the pushforward measure $f_*\mu_{T^n}$ which is a measure on N. Recall from Remark 1.1.21 (i) that the vanishing ideal associated to $f_*\mu_{T^n}$ is defined by

$$\mathbf{0}(A) := f_*\mu_{T^n}(N \setminus A) = \ker\left[H^*T^n \to H^*f^{-1}A\right]$$

for every closed subset $A \subseteq N$. Corollary 1.1.3 implies

$$\operatorname{rk}\left[H^1T^n \to H^1f^{-1}y\right] = \operatorname{rk}\left[H^1T^n \big/ \mathbf{0}(y) \cap H^1T^n\right]$$

for every $y \in N$ and therefore the condition $\operatorname{width}_1(f) < n-q$ translates into

$$\operatorname{rk}\left[H^1T^n \big/ \mathbf{0}(y) \cap H^1T^n\right] < n-q.$$

Choose an arbitrary metric on T^n. With respect to this metric we have

$$\bigcap_{m=1}^{\infty} \overline{B\left(y, \frac{1}{m}\right)} = \{y\}.$$

The continuity property of $f_*\mu_{T^n}$ (translated into the language of vanishing ideals) yields

$$\bigcup_{m=1}^{\infty} \mathbf{0}\left(\overline{B\left(y, \frac{1}{m}\right)}\right) = \mathbf{0}(\{y\}).$$

Since H^*T^n is finitely generated there exists an $m(y) \gg 0$ depending on y such that

$$\mathbf{0}\left(\overline{B\left(y, \frac{1}{m(y)}\right)}\right) = \mathbf{0}(\{y\}).$$

For every closed subset $A \subset B\left(y, \frac{1}{m(y)}\right)$ we have $\mathbf{0}(A) \supseteq \mathbf{0}\left(\overline{B\left(y, \frac{1}{m(y)}\right)}\right)$ and hence

$$\operatorname{rk}[H^1T^n \to H^1A] = \operatorname{rk}\left[H^1T^n \Big/ \mathbf{0}(A) \cap H^1T^n\right] \tag{2.1}$$

$$\leq \operatorname{rk}\left[H^1T^n \Bigg/ \mathbf{0}\left(\overline{B\left(y, \frac{1}{m(y)}\right)}\right) \cap H^1T^n\right] = \operatorname{rk}\left[H^1T^n \Big/ \mathbf{0}(y) \cap H^1T^n\right] < n - q. \tag{2.2}$$

Every continuous map f can be uniformly approximated by smooth maps g_m. Since M and N are compact and metrisable the upper semi-continuity of width_1 (cf. Proposition 2.2.2) implies $\operatorname{width}(g_m) \leq \operatorname{width}(f) < n - q$ for $m \gg 0$. So without loss of generality we can assume that f itself is smooth. Applying Proposition 1.11 to this smooth map $f\colon T^n \to N^q$ and the open cover $\left(B\left(y, \frac{1}{m(y)}\right)\right)_{y \in N}$ yields a smooth triangulation $\mathcal{T}$ of N and a sequence of smooth maps $f_m\colon M \to N$ uniformly converging to f such that the following two properties hold:

(i) Every map f_m intersects every simplex $\sigma \in \mathcal{T}_k$ stratum transversally.

(ii) For every simplex $\sigma \in \mathcal{T}_k$ there exists a $y(\sigma) \in N$ such that

$$\sigma(\Delta^k) \subseteq B\left(y(\sigma), \frac{1}{m(y(\sigma))}\right).$$

Using estimate (2.2) we conclude that $F_\sigma := f^{-1}\sigma(\Delta^k)$ satisfies $\operatorname{rk}\left[H^1T^n \to H^1F_\sigma\right] < n-q$. Similarly as before we can use the upper semi-continuity of width_1 (Proposition 2.2.2) to get $\operatorname{width}_1(f_m) < n - q$ for $m \geq M$. The map f_M contradicts Proposition 2.3. □

Remark 2.4. In the future whenever we want to prove a lower bound for cohomological waist we will reduce it to the proof of a statement similar to Proposition 2.3. We will not carry out this reduction in detail anymore and will simply say something along the following lines:

"Without loss of generality f is smooth and there exists a smooth triangulation $\mathcal{T}$ of N which is generic with respect to f and fine, i.e. the following properties hold:

(i) The map f intersects every simplex of $\mathcal{T}$ stratum transversally.

(ii) For every simplex σ the preimage $F_\sigma := f^{-1}\sigma(\Delta^k)$ satisfies $\operatorname{rk}\left[H^1(T^n;\mathbb{Z}) \to H^1(F_\sigma;\mathbb{Z})\right] < n - q$."

Later on we will not even repeat what "generic and fine" means and assume it will be clear from the context.

3 Cohomological filling

Recall that Theorem 2.1 we are trying to prove is about a map $f\colon T^n \to N^q$ and its fibers $F_y := f^{-1}y$. Amongst others we want to apply the following statements to the inclusions of the fibers $f_y\colon F_y \hookrightarrow T^n$. The reader shall bear this example in mind.

Motivation 3.1. For every continuous map $k\colon K \to T^n$ satisfying $\operatorname{rk} H^1(k;\mathbb{Z}) < n-q$ the induced maps $H^j(k;\mathbb{Z})\colon H^j(T^n;\mathbb{Z}) \to H^jK$ vanish for $j \geq n-q$.

Proof. It suffices to prove the case $j = n-q$. Consider an arbitrary monomial of degree $n-q$, without loss of generality $x_1 \cdot \dots \cdot x_{n-q}$. Since $\operatorname{rk} H^1(k;\mathbb{Z}) < n-q$ there exists one factor, without loss of generality x_{n-q}, such that k^*x_{n-q} can be expressed as

$$k^*x_{n-q} = \sum_{i<n-q} \lambda_i k^* x_i$$

and hence

$$\begin{aligned} k^*(x_1 \cdot \ldots \cdot x_{n-q}) &= (k^*x_1) \cdot \ldots \cdot (k^*x_{n-q}) \\ &= (k^*x_1) \cdot \ldots \cdot (k^*x_{n-q-1}) \cdot \sum_{i<n-q} \lambda_i (k^*x_i) = 0. \quad \square \end{aligned}$$

This motivates the following

Lemma 3.2 (Filling Lemma). Let $k\colon K \to T^n$ be continuous and $\operatorname{rk} H^1(k;\mathbb{Z}) < n-q$. There exists a relative CW complex $(\mathrm{Fill}(k), K)$ and an extension $\mathrm{fill}(k)\colon \mathrm{Fill}(k) \to T^n$ such that the diagram

$$\begin{array}{ccc} \mathrm{Fill}(k) & & \\ \iota \uparrow & \searrow^{\mathrm{fill}(k)} & \\ K & \xrightarrow{\;k\;} & T^n \end{array}$$

commutes and the following properties hold.

(i) Up to homotopy $\mathrm{Fill}(k)$ is the disjoint sum of a number of tori, one copy for each component of K, i.e.

$$\mathrm{Fill}(k) \simeq T^{r_1} \amalg T^{r_2} \amalg \dots$$

and the dimensions satisfy $r_i < n-q$. In particular we have $H_{\geq n-q}(\mathrm{Fill}(k);G) = 0$ and $H_{\geq n-q}(\iota;G) = 0$ for any abelian coefficient group G.

(ii) $(\mathrm{Fill}(k), K)$ is 1-connected

(iii) $\operatorname{rk} H^1(\mathrm{fill}(k);\mathbb{Z}) = \operatorname{rk} H^1(k;\mathbb{Z})$

Before we prove the lemma we need an analysis of the discrepancy between cohomology and homology.

Remark 3.3. (i) For every continuous map $f\colon X \to Y$ we have $\operatorname{rk} H_1(f;\mathbb{Z}) = \operatorname{rk} H^1(f;\mathbb{Z})$.

(ii) If $H_1(f;\mathbb{Z})$ is an isomorphism so is $H^1(f;\mathbb{Z})$.

Proof. (i) The cohomology group H^1X is determined by the short exact sequence

$$0 \to \operatorname{Ext}(H_0(X;\mathbb{Z}),\mathbb{Z}) \to H^1(X;\mathbb{Z}) \to \operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

and since $H_0(X;\mathbb{Z})$ is free the Ext-term vanishes and we have an isomorphism

$$H^1(X;\mathbb{Z}) \xrightarrow{\cong} \operatorname{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}).$$

Naturality yields the commutativity of the diagram

$$\begin{array}{ccc} H^1(Y;\mathbb{Z}) & \xrightarrow{\cong} & \operatorname{Hom}(H_1(Y;\mathbb{Z}),\mathbb{Z}) \\ \downarrow{\scriptstyle H^1 f} & & \downarrow{\scriptstyle \operatorname{Hom}(H_1 f,\mathbb{Z})} \\ H^1(X;\mathbb{Z}) & \xrightarrow[\cong]{} & \operatorname{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}). \end{array} \tag{3.1}$$

One way to define the rank of a linear map $\varphi\colon A \to B$ between abelian groups A and B is to set

$$\operatorname{rk}\varphi := \operatorname{rk}(\varphi \otimes \mathbb{Q}).$$

But there are natural maps

$$\operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q},\mathbb{Q})$$

which are isomorphisms of $\mathbb{Q}$-vector spaces, i.e. dualising and rationalising commute up to natural equivalence. This proves

$$\operatorname{rk}\operatorname{Hom}(H_1 f,\mathbb{Z}) = \operatorname{rk} H_1 f.$$

(ii) Since $\operatorname{Hom}(-,\mathbb{Z})$ turns isomorphism into isomorphism so we get that $\operatorname{Hom}(H_1(f;\mathbb{Z});\mathbb{Z})$ is an isomorphism. Together with diagram (3.1) we conclude that $H^1(f;\mathbb{Z})$ is an isomorphism.

□

We will frequently change our point of view between cohomology and homology and we will do so without further reference to the remark above.

Notation 3.4. From now on we will have to introduce a lot of spaces all of which come with reference maps to T^n. As with $f_y\colon F_y \to T^n$ these reference maps are denoted by the lower case letters corresponding to the upper case letters representing the spaces.

Proof of Filling Lemma 3.2. Let us first discuss the case where K is connected and let $r := \operatorname{rk} H^1(k;\mathbb{Z})$. By the naturality of the Hurewicz homomorphism the following diagram commutes.

$$\begin{array}{ccc} \pi_1 K & \longrightarrow & \pi_1 T^n = \mathbb{Z}^n \\ \downarrow & & \downarrow \cong \\ H_1(K;\mathbb{Z}) & \longrightarrow & H_1(T^n;\mathbb{Z}) = \mathbb{Z}^n \end{array} \tag{3.2}$$

This proves that $\operatorname{im} \pi_1 k \subseteq \mathbb{Z}^n$ is also a rank r subgroup. Consider a covering $T^r \times \mathbb{R}^{n-r} \to T^n$ corresponding to this subgroup. Hence there exists a lift $\widetilde{k}\colon K \to T^r \times \mathbb{R}^{n-r}$ such that

$$\begin{array}{ccc} & \overset{\widetilde{k}}{\nearrow} & T^r \times \mathbb{R}^{n-r} \\ & & \downarrow \\ K & \xrightarrow{\;k\;} & T^n \end{array}$$

commutes. On the level of fundamental groups this turns into the following diagram.

$$\begin{array}{ccc} & \overset{\pi_1\widetilde{k}}{\nearrow} & \pi_1\left(T^r \times \mathbb{R}^{n-r}\right) \\ & & \downarrow \\ \pi_1 K & \xrightarrow{\;\pi_1 k\;} & \pi_1 T^n \end{array}$$

where (by construction of the covering) the vertical arrow is the inclusion $\operatorname{im} \pi_1 k \subset \mathbb{Z}^n$. Thus $\pi_1\widetilde{k}$ is obtained from $\pi_1 k$ by restricting the target to $\operatorname{im} \pi_1 k$, in particular $\pi_1\widetilde{k}$ is surjective. Using the naturality of the Hurewicz homomorphism similar to (3.2) we conclude that $H_1(\widetilde{k};\mathbb{Z})$ is surjective.

We want to turn $\widetilde{k}$ into the inclusion of relative CW complex. Substitute $T^r \times \mathbb{R}^{n-r}$ by the mapping cylinder $M_{\widetilde{k}}$ and choose a relative CW approximation $(\operatorname{Fill}(k), K)$ of $(M_{\widetilde{k}}, K)$, i.e. there is a weak homotopy equivalence $\operatorname{Fill}(k) \xrightarrow{\simeq} M_{\widetilde{k}}$ restricting to the identity on K. Define ι and $\operatorname{fill}(k)$ as in the following diagram.

$$\begin{array}{ccccc} \operatorname{Fill}(k) & \xrightarrow{\simeq} & M_{\widetilde{k}} & \longrightarrow & T^n \\ & \nwarrow^{\iota} & \uparrow & \nearrow_{k} & \\ & & K & & \end{array} \qquad \operatorname{fill}(k)\colon \operatorname{Fill}(k) \to T^n$$

The induced map $H_*(\iota;\mathbb{Z})$ is an isomorphism for $* = 0$ and surjective for $* = 1$ since $H_*(\widetilde{k};\mathbb{Z})$ has these properties. This implies that $(\operatorname{Fill}(k), K)$ is 1-connected. The surjectivity of $H_1(\iota;\mathbb{Z})$ also implies $\operatorname{im} H_1(\operatorname{fill}(k);\mathbb{Z}) = \operatorname{im} H_1(k;\mathbb{Z})$ and together with Remark 3.3 (i) we get property (iii). If K is not connected we can apply the construction above to all of its components. □

Remark 3.5. (i) Using obstruction theory one can show that the extension $\mathrm{fill}(k)$ is unique up to homotopy relative to K but we will not need this.

(ii) CW approximation also shows that the relative CW complex $(\mathrm{Fill}(k), K)$ has only attaching cells of dimension at least 2. Plenty of the later constructions could be considerably simplified by keeping this in mind. But in a later section we can only generalise the argument using some high connectivity. Thus we will start to use it from the beginning.

(iii) Observe that in the lemma above it is important to assume that the rank $\mathrm{rk}\, H^1(k; \mathbb{Z})$ is measured with coefficients in $\mathbb{Z}$. This is due to the usage of the Hurewicz theorem and covering space theory. There is no simple analogue to Filling Lemma with coefficients in $\mathbb{Z}_2$ since e.g. the double cover map $k\colon S^1 \to S^1$ satisfies $\mathrm{rk}\, H^1(k; \mathbb{Z}_2) = 0$ but cannot be filled.

Actually we could finish the proof of Theorem 2.1 right now but we want to introduce the language of *cycle spaces* which offer a more conceptual viewpoint.

4 The space of cycles

In this section two kinds of chain complexes will appear, namely singular and the simplicial chain complexes and it should always be clear from the context which one we mean depending on whether we apply it to topological spaces or simplicial sets. Nevertheless in order to avoid confusion we will consistently try to denote the singular chain complex by C_* and the simplicial chain complex by $C_\bullet$.

Let $f\colon M^n \to N^q$ be a smooth map between closed R-oriented manifolds, σ a smooth embedded k-simplex in N which intersects f stratum transversally. Recall Proposition 1.8 by which we can assign to every vertex v of σ an $(n-q)$-cycle c_v in M and to any l-dimensional face τ of σ an $(n-q+l)$-chain c_τ such that we have

$$\partial c_\tau = \sum_{i=0}^{l} c_{\partial_i \tau}.$$

This motivates the following

Definition 4.1. Let (D_*, ∂) be a chain complex of abelian groups. The *space of* $(n-q)$*-cycles in* D_* is a simplicial set denoted by $cl^{n-q}(D_*, \partial)$ the level sets of which are given by

$$\left(cl^{n-q}(D_*, \partial)\right)_k := \left(cl^{n-q} D_*\right)_k := \mathrm{Hom}\left(C_\bullet \Delta[k], D_{*+(n-q)}\right).$$

Some explanations are in order.

(i) $\Delta[k]$ denotes the k-dimensional standard simplex in the category **sSet**.

(ii) $C_\bullet\Delta[k]$ denotes its normalised chain complex, i.e. the chain groups are generated only by the non-degenerate simplices of $\Delta[k]$.

(iii) The Hom set is meant as the set of morphisms of chain complexes of abelian groups.

The right hand side defines a contravariant functor $\Delta \to \mathbf{Set}$ where Δ is the ordinal number category. This turns $cl^{n-q}D_*$ into a simplicial set.

The main example of a chain complex D_* to which we want to apply the construction above is the singular chain complex of the source manifold, e.g. a torus.

Remark 4.2. (i) The chain groups $C_i\Delta[k]$ are non-zero only for $0 \leq i \leq k$ and $C_k\Delta[k] \cong \mathbb{Z}$. Let $c_k \in C_k\Delta[k]$ be a generator.

(ii) Thus a 0-simplex σ in $cl^{n-q}D_*$ corresponds to a diagram of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & C_0\Delta[0] & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \sigma_0} & & \downarrow \\ D_{(n-q)+1} & \longrightarrow & D_{n-q} & \longrightarrow & D_{(n-q)-1}. \end{array}$$

This diagram is uniquely determined by the image $\sigma_0 c_0 \in C_{n-2}$ and this element satisfies $\partial\sigma_0 c_0 = 0$. So 0-simplices are in bijection to $(n-q)$-cocycles of D_*.

(iii) Any k-simplex σ of $cl^{n-q}D_*$ is precisely a diagram of the following form.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_k\Delta[k] & \longrightarrow & \dots & \longrightarrow & C_0\Delta[k] & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \sigma_k} & & & & \downarrow{\scriptstyle \sigma_0} & & \downarrow \\ D_{(n-q)+k+1} & \longrightarrow & D_{(n-q)+k} & \longrightarrow & \dots & \longrightarrow & D_{n-q} & \longrightarrow & D_{(n-q)-1}. \end{array}$$

(iv) The simplicial set $cl^{n-q}D_*$ depends covariantly on the chain complex argument D_*, turning cl^{n-q} into a covariant functor $cl^{n-q}\colon \mathbf{ChainCom} \to \mathbf{sSet}$.

We need some preparation in order to rigorously prove that – like we tried to motivate – Proposition 1.8 yields elements in $(cl^{n-q}C_*M)_k$.

Lemma 4.3. For any $\sigma \in (cl^{n-q}D_*)_k$ there is an evaluation map

$$\mathrm{ev}_k\colon \left(cl^{n-q}D_*\right)_k \to D_{(n-q)+k}$$

and these extend and fit together such that

$$\mathrm{ev}_\bullet\colon C_\bullet cl^{n-q}D_* \to D_{\bullet+(n-q)}$$

is a morphism of chain complexes. The source of ev_* is the simplicial chain complex of $cl^{n-q}D_*$. Sometimes we abbreviate $\mathrm{ev}_k\sigma$ by $\widehat{\sigma}$.

Proof. We have to check the commutativity of

$$
\begin{array}{ccc}
C_k cl^{n-q} D_* & \xrightarrow{\mathrm{ev}_k} & D_{(n-q)+k} \\
{\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial} \\
C_{k-1} cl^{n-q} D_* & \xrightarrow[\mathrm{ev}_{k-1}]{} & D_{(n-q)+k-1}.
\end{array}
$$

and this is equivalent to

$$\mathrm{ev}_{k-1}\partial\sigma = \partial \mathrm{ev}_k \sigma$$

for every $\sigma \in (cl^{n-q}D_*)_k$. We have

$$
\begin{aligned}
\mathrm{ev}_{k-1}\partial\sigma &= \sum_i (-1)^i \partial_i \sigma \\
&= \sum_i (-1)^i \mathrm{ev}_{k-1}\partial_i \sigma.
\end{aligned}
$$

The faces $\partial_i\sigma \in (cl^{n-q}D_*)_k$ are given by the following precomposition with the inclusion of faces $d_i\colon \Delta[k-1] \to \Delta[k]$.

$$
\begin{array}{ccccccccccc}
 & & & & C_{k-1}\Delta[k-1] & \longrightarrow & \dots & \longrightarrow & C_0\Delta[k-1] & & \\
 & & & & {\scriptstyle C_{k-1}d_i}\downarrow & & & & \downarrow & & \\
0 & \longrightarrow & C_k\Delta[k] & \longrightarrow & C_{k-1}\Delta[k] & \longrightarrow & \dots & \longrightarrow & C_0\Delta[k] & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle \sigma_k}\downarrow & & \downarrow & & & & \downarrow{\scriptstyle \sigma_0} & & \downarrow \\
D_{(n-q)+k+1} & \longrightarrow & D_{(n-q)+k} & \longrightarrow & D_{(n-q)+k-1} & \longrightarrow & \dots & \longrightarrow & D_{n-q} & \longrightarrow & C_{(n-q)-1}.
\end{array}
\tag{4.1}
$$

From this we can continue

$$
\begin{aligned}
\mathrm{ev}_{k-1}\partial\sigma &= \sum_i (-1)^i \mathrm{ev}_{k-1}\partial_i \sigma \\
&= \sum_i (-1)^i \sigma_{k-1}\partial_i c_k = \sigma_{k-1}\partial c_k = \partial\sigma_k c_k = \partial \mathrm{ev}_k \sigma. \quad \square
\end{aligned}
$$

The following lemma shows how $(k+1)$ simplices $\varphi_i \in (cl^{n-q}D_*)_k$ can be glued together to form the faces of a simplex $\sigma \in (cl^{n-q}D_*)_{k+1}$ if the obvious homological restriction in D_* vanishes.

Lemma 4.4 (Gluing Lemma)**.** Let $\varphi_0, \dots, \varphi_{k+1} \in (cl^{n-q}D_*)_k$ such that

$$\partial_i \varphi_j = \partial_{j-1}\varphi_i, \quad 0 \le i < j \le k+1.$$

If there exists an element $\overline{\sigma} \in D_{(n-q)+k+1}$ with

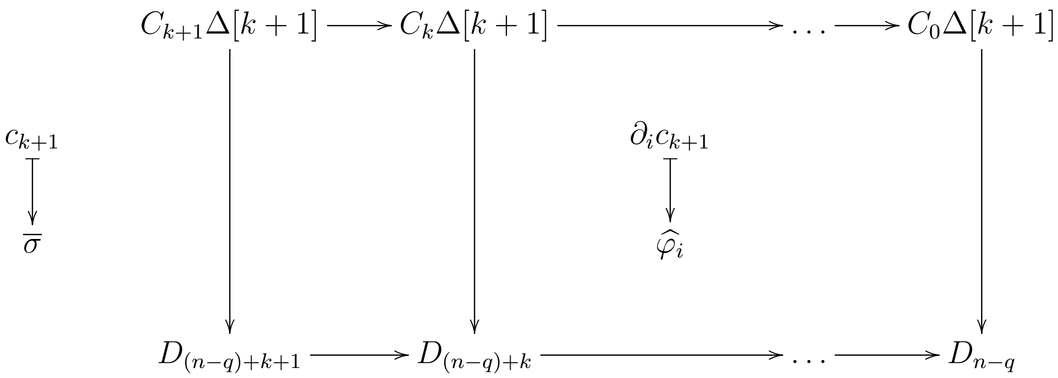
$$\partial \overline{\sigma} = \sum_{i=0}^{k+1} (-1)^i \widehat{\varphi}_i$$

there is a unique $\sigma \in \left(cl^{n-q} D_*\right)_{k+1}$ satisfying $\widehat{\sigma} = \overline{\sigma}$ and $\partial_i \sigma = \varphi_i$.

Proof. For every $\alpha \in cl^{n-q} D_*$ diagram (4.1) implies the identity

$$\partial \widehat{\alpha} = \sum_{i=0}^{l} (-1)^i \widehat{\partial_i \alpha} \tag{4.2}$$

Given the element $\overline{\sigma}$ we can consider the following diagram.



All the lower dimensional simplices are of the form $\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}$ and they shall be mapped such that

$$\sigma : \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1} \mapsto \left(\partial_{i_1} \dots \partial_{i_l} \varphi_i\right)^\wedge$$

This image is indeed invariant under applying simplicial identities to the non-unique representation $\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}$ and we get

$$\begin{aligned} & \sigma\left(\partial \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}\right) \\ =& \sigma\left(\sum_{m=0}^{k-l} (-1)^m \partial_m \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}\right) \\ =& \sum_{m=0}^{k-l} (-1)^m \left(\partial_m \partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}\right)^\wedge \\ =& \partial \left(\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}\right)^\wedge = \partial \sigma\left(\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}\right) \end{aligned}$$

where in the passage to the last line we have applied Lemma 4.3 to $\partial_{i_1} \dots \partial_{i_l} \partial_i c_{k+1}$. Thus σ commutes with the differentials. $\square$

Construction 4.5. Recall Proposition 1.8. Let $f\colon M^n \to N^q$ be a smooth map between closed R-oriented manifolds, $\mathcal{T}$ an R-oriented triangulation of N such that f intersects all the simplices $\sigma \in \mathcal{T}_q$ stratum transversally. We can assign to any $\sigma \in \mathcal{T}_k$ a singular chain $c_\sigma \in C_{n-q+k}(F_\sigma; R)$ such that the following properties hold:

(i) For $\sigma \in \mathcal{T}_0$ the chain $c_\sigma \in C_{n-q}(F_\sigma; R)$ represents the (correctly oriented) fundamental class of F_σ.

(ii) For $1 \leq k \leq q$ and $\sigma \in \mathcal{T}_k$ we have

$$\partial c_\sigma = \sum_{i=0}^{k} (-1)^i c_{\partial_i \sigma}$$

as an equation in $C_{n-q+k-1}(F_\sigma; R)$ and c_σ represents the (correctly oriented) relative fundamental class in $H_{n-q+k}(F_\sigma, \partial F_\sigma; R)$.

(iii) The sum

$$\sum_{\sigma \in \mathcal{T}_q} c_\sigma \in C_n(M; R)$$

represents the (correctly oriented) fundamental class of M.

For every $\sigma \in \mathcal{T}_0$ we can use Remark 4.2 (ii) to turn the cycles c_σ into 0-simplices $z_\sigma \in (cl^{n-q}C_*(F_\sigma; R))_0$ satisfying $\widehat{z_\sigma} = c_\sigma$.

For higher-dimensional $\sigma \in \mathcal{T}_k$ we will inductively construct elements $z_\sigma \in (cl^{n-q}C_*(F_\sigma; R))_k$ satisfying

$$\partial_i z_\sigma = z_{\partial_i \sigma} \quad \text{and} \quad \widehat{z_\sigma} = c_\sigma. \tag{4.3}$$

Assume we have constructed such simplices z_τ for all τ of dimension at most k and fix $\sigma \in \mathcal{T}_{k+1}$. For $0 \leq i < j \leq k+1$ we have

$$\partial_i z_{\partial_j \sigma} = z_{\partial_i \partial_j \sigma} = z_{\partial_{j-1} \partial_i \sigma} = \partial_{j-1} z_{\partial_i \sigma}$$

and

$$\partial c_\sigma = \sum_{i=0}^{k+1} (-1)^i c_{\partial_i \sigma} = \sum_{i=0}^{k+1} (-1)^i \widehat{z_{\partial_i \sigma}}.$$

Applying the Gluing Lemma 4.4 to $\varphi_i := z_{\partial_i \sigma}$ and $\overline{\sigma} := c_\sigma$ yields a simplex $z_\sigma \in (cl^{n-q}C_*(F_\sigma; R))_{k+1}$ with the desired properties (4.3).

The simplicial chain

$$Z(f, \mathcal{T}) := \sum_{\sigma \in \mathcal{T}_q} z_\sigma$$

can be viewed as an element in $C_q cl^{n-q}C_*(M; R)$ and it satisfies $\partial Z(f, \mathcal{T}) = 0$. Since

$$\mathrm{ev}_\bullet \colon C_\bullet cl^{n-q}C_*(M; R) \to D_{\bullet+(n-q)}$$

is a morphism of chain complexes and maps $Z(f, \mathcal{T})$ to $\sum_{\sigma \in \mathcal{T}_q} c_\sigma$ we conclude that $[Z(f, \mathcal{T})] \neq 0$ in $H_q cl^{n-q} C_*(M; R)$.

Comment 4.6. (i) There is an analytic analogue to the construction above. Let $M^n \subset \mathbb{R}^N$ be a smooth closed embedded manifold, $I_k(M)$ be the topological space of integral currents with the flat topology and $Z_k(M) \subset I_{n-q}(M)$ the subspace of cycles. In [Alm62] Almgren proved that the homotopy groups of the latter are given by

$$\pi_i Z_k(M) \cong H_{i+k}(M).$$

A priori the homotopy groups of a space do not determine its homotopy type since it could have non-zero k-invariants but in the case of $Z_k(M)$ the topological group completion theorem implies that the k-invariants of every topological abelian monoid vanish. In particular we get

$$Z_k(M) \simeq \prod_{i=0}^{n-k} K(H_{i+k}(M), i). \tag{4.4}$$

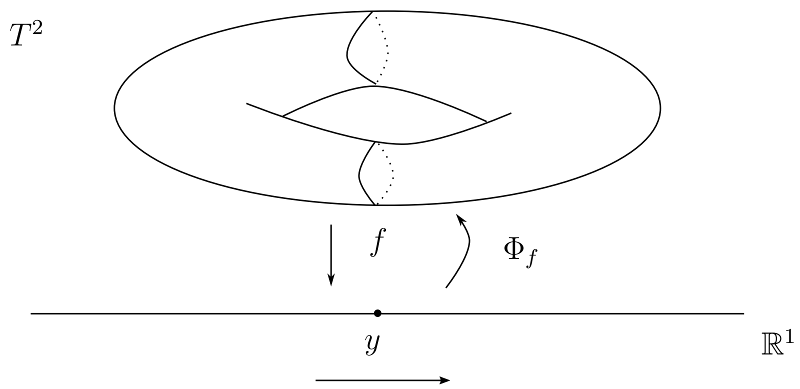
One reasonable corollary from this is $\pi_0 Z_k(M) = H_k M$. Another consequence is that

$$\pi_q Z_{n-q}(M) \cong H_n(M) \cong \mathbb{Z} \tag{4.5}$$

with the generator given as follows. Let $f\colon M \subset \mathbb{R}^N \to \mathbb{R}^q$ be a generic projection. For any $y \in \mathbb{R}^q$ the preimage $f^{-1}(q)$ defines an $(n-q)$-dimensional integral cycle and the map

$$\begin{aligned} \Phi_f\colon \mathbb{R}^q &\to Z_{n-q}(M) \\ y &\mapsto f^{-1}(y) \end{aligned}$$

is continuous and maps everything outside of $\operatorname{im} f$ to the zero cycle. Hence it determines an element $[\Phi_f] \in \pi_q Z_{n-q}(M)$ which is independent of f and corresponds exactly to the fundamental class under the correspondence (4.5).



It is an important observation – especially when proving waist inequalities – that every map $f\colon M^n \to \mathbb{R}^q$ yields a homotopically nontrivial map $\Phi_f\colon \mathbb{R}^q \to Z_{n-q}(M)$. There are different ways to formalise the notion of spaces of cycles. For obvious reasons we chose a definition with the flavour of algebraic topology.

(ii) Of course one wonders what is the homotopy type of $cl^{n-q}D_*$ for a given chain complex D_*. Up to an index shift cl^{n-q} is just the Dold-Kan correspondence between chain complexes and simplicial abelian groups and from that we get in analogy to (4.4)

$$cl^{n-q}D_* \simeq \prod_{i=0}^{\infty} K(H_{n-q+i}(D_*), i).$$

(iii) In the construction above the cycle $Z(f, \mathcal{T})$ in $cl^{n-q}C_*(M; R)$ is called the *canonical cycle associated to f and $\mathcal{T}$* and $[Z(f; \mathcal{T})] \in H_q cl^{n-q}C_*(M; R)$ the *canonical homology class.* The cycle $Z(f, \mathcal{T})$ depends heavily on the map f and the triangulation $\mathcal{T}$ whereas one can show that $[Z(f; \mathcal{T})]$ is independent of these choices. We could define the canonical homology class far easier as being represented by the q-simplex given by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_q\Delta[q] & \longrightarrow & \dots & \longrightarrow & C_0\Delta[q] & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \sigma_q} & & & & \downarrow{\scriptstyle \sigma_0} & & \downarrow \\
C_{n+1}M & \longrightarrow & C_nM & \longrightarrow & \dots & \longrightarrow & C_{n-q}M & \longrightarrow & D_{(n-q)-1}.
\end{array}$$

where σ_q maps c_q to a fundamental cycle of M and all other c_i vanish. This cycle arises from the geometric construction above if there exists one large q-simplex containing the $\operatorname{im} f$.

However this cycle does not incorporate the map f and the fine triangulation $\mathcal{T}$ in such a way which will enable us to execute the proof of Proposition 2.3 which we restate for convenience.

Proposition 2.3. Let $f\colon T^n \to N^q$ be a smooth map where N is a closed manifold together with a smooth triangulation $\mathcal{T}$ the simplices of which intersect f stratum transversally. Then there exists a simplex $\sigma \in \mathcal{T}_k$ such that the preimage $F_\sigma := f^{-1}\sigma(\Delta^k)$ satisfies

$$\operatorname{rk}\left[H^1(T^n; \mathbb{Z}) \to H^1(F_\sigma; \mathbb{Z})\right] \geq n - q.$$

In the following proof there will be a certain unpleasant mixture of coefficients between $\mathbb{Z}$ and $\mathbb{Z}_2$. After all this could not have been totally avoided since we do not want to assume the target manifold N to be orientable which introduces $\mathbb{Z}_2$ coefficients at some places. On the other hand, as explained in Remark 3.5 (iii), the usage of Filling Lemma 3.2 forces us to interpret some expressions, e.g. (**??**), with coefficients in $\mathbb{Z}$.

Proof of Proposition 2.3. We proceed by contradiction and assume that such a map f and triangulation $\mathcal{T}$ exist. Recall the simplices $z_\sigma \in (cl^{n-q}C_*(F_\sigma;\mathbb{Z}_2))_k$ and the canonical cycle

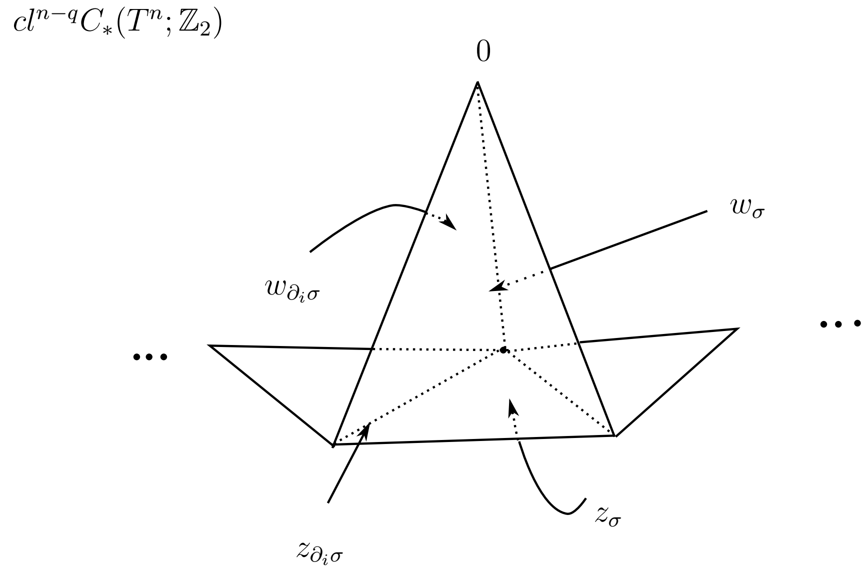
$$Z(f;\mathcal{T}) := \sum_{\sigma\in\mathcal{T}_q} z_\sigma \in C_q cl^{n-q}C_*(T^n;\mathbb{Z}_2)$$

from Construction 4.5.

We will build the cone of Z inside $cl^{n-q}C_*(T^n;\mathbb{Z}_2)$. For every $\sigma \in \mathcal{T}_k$ we will construct simplices $w_\sigma \in (cl^{n-q}C_*(T^n;\mathbb{Z}_2))_{k+1}$ satisfying

$$\partial_i w_\sigma = \begin{cases} w_{\partial_i\sigma} & \text{if } 0 \le i \le k \\ z_\sigma & \text{if } i = k+1. \end{cases} \tag{4.6}$$

For $\sigma \in \mathcal{T}_0$ and $i = 0$ equation (4.6) shall be interpreted as $\partial_0 w_\sigma = w_{\partial_0\sigma} = 0$.



If we constructed such simplices w_σ the standard cone calculation shows

$$\partial \sum_{\sigma\in\mathcal{T}_q} w_\sigma = (-1)^{q+1} Z(f;\mathcal{T})$$

contradicting Construction 4.5 where we have seen that $[Z(f;\mathcal{T})] \neq 0$ in $H_q cl^{n-q}C_*(T^n;\mathbb{Z}_2)$. So we are only left with constructing simplices w_σ satisfying equation (4.6).

Recall Notation 3.4 that every map from a topological space to T^n is denoted by the lower case letter corresponding to the upper case letter representing the space. For every $0 \le k \le q$ and $\sigma \in \mathcal{T}_k$ we will inductively construct triples $(L_\sigma, K_\sigma, F_\sigma)$ of topological spaces and simplices $w_\sigma \in (cl^{n-q}C_*(L_\sigma;\mathbb{Z}_2))_{k+1}$ such that the following properties hold.

(i) (L_σ, F_σ) is a 1-connected relative CW complex and we write

$$L_\sigma = F_\sigma \cup e_\sigma \tag{4.7}$$

where e_σ is an abbreviation for all the cells which we need to attach to F_σ in order to obtain L_σ.

(ii) There are canonical inclusions as in the following diagram.

$$\begin{array}{ccc} L_{\partial_i\sigma} & \hookrightarrow & L_\sigma \\ \uparrow & \searrow & \uparrow \\ K_{\partial_i\sigma} & \hookrightarrow & K_\sigma \\ \uparrow & & \uparrow \\ F_{\partial_i\sigma} & \hookrightarrow & F_\sigma \end{array}$$

(iii) There exist extensions such that the diagram

$$\begin{array}{ccc} L_\sigma & & \\ \uparrow & \overset{l_\sigma}{\searrow} & \\ K_\sigma & \xrightarrow{k_\sigma} & T^n \\ \uparrow & \underset{f_\sigma}{\nearrow} & \\ F_\sigma & & \end{array}$$

commutes.

(iv) $\operatorname{rk} H^1(l_\sigma;\mathbb{Z}) = \operatorname{rk} H^1(k_\sigma;\mathbb{Z}) = \operatorname{rk} H^1(f_\sigma;\mathbb{Z}) < n-q$

(v) We have $H_{\geq n-q}(L_\sigma;\mathbb{Z}_2) = 0$ and in particular $H_*(K_\sigma;\mathbb{Z}_2) \to H_*(L_\sigma;\mathbb{Z}_2)$ for $* \geq n-q$.

(vi) The simplices w_σ satisfy (4.6) as a relation of simplices $cl^{n-q}C_*(L_\sigma;\mathbb{Z}_2)$. Naturally it can also be seen as a relation in $cl^{n-q}C_*(T^n;\mathbb{Z}_2)$.

In the base case $k=0$ we can set $K_\sigma := F_\sigma$. By assumption we have $\operatorname{rk} H^1(f_\sigma;\mathbb{Z}) < n-q$ and we can apply Filling Lemma 3.2 to it. We get a relative CW complex (L_σ, F_σ) and an extension

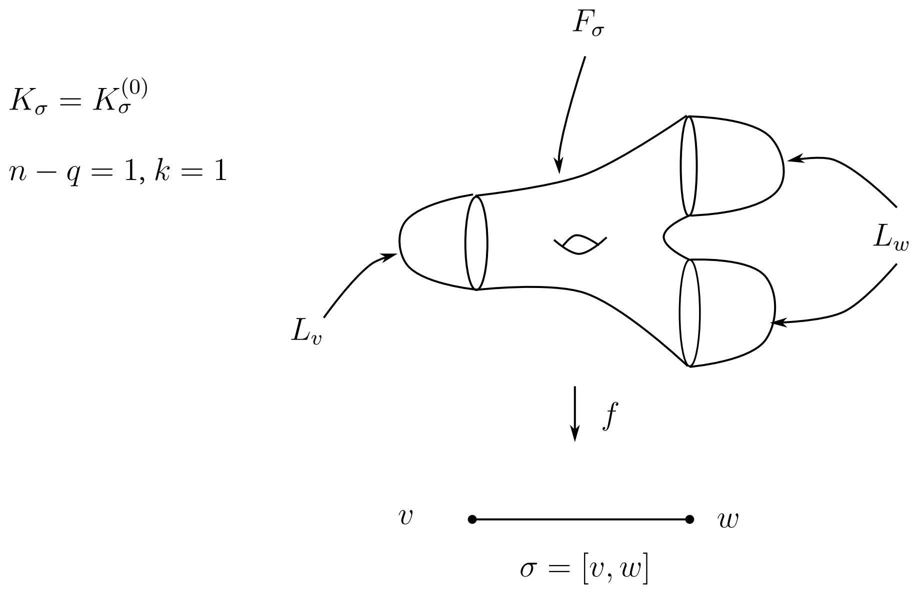
$$\begin{array}{ccc} L_\sigma & & \\ \uparrow & \overset{l_\sigma}{\searrow} & \\ K_\sigma & \longrightarrow & T^n \\ \mathrm{id}\,\Vert & \underset{f_\sigma}{\nearrow} & \\ F_\sigma & & \end{array} \tag{4.8}$$

satisfying (iv). Consider the cycle $\widehat{z_\sigma} \in C_{n-q}(F_\sigma;\mathbb{Z})$ and its image under the inclusion $F_\sigma = K_\sigma \hookrightarrow L_\sigma$. Since $H_{n-q}(L_\sigma;\mathbb{Z}_2) = 0$ there exists a (suggestively denoted) chain $\widehat{w_\sigma} \in C_{n-q+1}(L_\sigma;\mathbb{Z}_2)$ such that

$$\partial\widehat{w_\sigma} = \widehat{z_\sigma}. \tag{4.9}$$

Using the Gluing Lemma 4.4 we get a simplex $w_\sigma \in \left(cl^{n-q}C_*(L_\sigma;\mathbb{Z}_2)\right)_1$ satisfying (4.6) for $k = 0$.

Assume K_τ, L_τ and w_τ have already been constructed for all simplices τ of dimension strictly less than $k \geq 1$.



For $\sigma \in \mathcal{T}_k$ and $0 \leq i < k$ we inductively define spaces and maps $k_\sigma^{(i)} \colon K_\sigma^{(i)} \to T^n$ by setting $K_\sigma^{(-1)} := F_\sigma$, $k_\sigma^{(-1)} := f_\sigma$ and

$$\begin{array}{ccc} K_\sigma^{(i-1)} \cup \bigcup_{i\text{-dim. faces } \tau \text{ of } \sigma} e_\tau =: K_\sigma^{(i)} & \xrightarrow{k_\sigma^{(i)}} & T^n \\ \uparrow & \nearrow_{k_\sigma^{(i-1)}} & \\ K_\sigma^{(i-1)} & & \end{array}$$

where we used the notation introduced in equation (4.7). This is well-defined since the targets of the attaching maps of e_τ are F_τ which canonically are subspaces of $F_\sigma \subseteq K_\sigma^{(i-1)}$ for every i.

We have a homeomorphism

$$\bigvee_{i\text{-dim. faces } \tau \text{ of } \sigma} \left(L_{\partial_i\sigma} \big/ F_{\partial_i\sigma} \right) \xrightarrow{\cong} K_\sigma^{(i)} \big/ K_\sigma^{(i-1)}.$$

Since the pairs $(L_{\partial_i\sigma}, F_{\partial_i\sigma})$ are 1-connected we conclude

$$H_*\left(K_\sigma^{(i)}, K_\sigma^{(i-1)}\right) \cong \bigoplus_{i\text{-dim. faces } \tau \text{ of } \sigma} H_*\left(L_{\partial_i\sigma}, F_{\partial_i\sigma}\right) = 0$$

for $* = 0, 1$ proving that the $\left(K_\sigma^{(i)}, K_\sigma^{(i-1)}\right)$ are 1-connected.

Let $K_\sigma := K_\sigma^{(k-1)}$, $k_\sigma := k_\sigma^{(k-1)}$. Since all the $\left(K_\sigma^{(i)}, K_\sigma^{(i-1)}\right)$ are 1-connected the same holds for (K_σ, F_σ). In particular the inclusion $F_\sigma \hookrightarrow K_\sigma$ induces a surjective homomorphism $H_1(F_\sigma; \mathbb{Z}) \to H_1(K_\sigma; \mathbb{Z})$. This surjectivity, Remark 3.3 (i) and the diagram

$$\begin{array}{ccc} K_\sigma & \xrightarrow{k_\sigma} & T^n \\ \uparrow & \nearrow_{f_\sigma} & \\ F_\sigma & & \end{array}$$

show that $\operatorname{rk} H^1(k_\sigma; \mathbb{Z}) = \operatorname{rk} H_1(k_\sigma; \mathbb{Z}) = \operatorname{rk} H_1(f_\sigma; \mathbb{Z}) = \operatorname{rk} H^1(f_\sigma; \mathbb{Z})$.

In particular we have $\operatorname{rk} H^1(k_\sigma; \mathbb{Z}) < n - q$ and we can apply Filling Lemma 3.2 to it in order to obtain the space $L_\sigma := \operatorname{Fill}(k_\sigma)$ and the map $l_\sigma := \operatorname{fill}(\sigma)$ satisfying (iii). The pair (L_σ, K_σ) is 1-connected and with the same calculation as above we get $\operatorname{rk} H^1(l_\sigma; \mathbb{Z}) = \operatorname{rk} H^1(k_\sigma; \mathbb{Z})$.

Using the inclusions $L_{\partial_i\sigma} \subseteq K_\sigma$ and $F_\sigma \subseteq L_\sigma$ we can consider the chain

$$y_\sigma := \sum_{i=0}^{k} (-1)^i \widehat{w_{\partial_i\sigma}} + (-1)^{k+1} \widehat{z_\sigma} \in C_{n-q+k}(K_\sigma; \mathbb{Z}_2). \tag{4.10}$$

Since $\partial y_\sigma = 0$ and $H_{n-q+k}(L_\sigma; \mathbb{Z}_2) = 0$ there exists a (suggestively denoted) chain $\widehat{w_\sigma} \in C_{n-q+k+1}(L_\sigma; \mathbb{Z}_2)$ satisfying $\partial \widehat{w_\sigma} = y_\sigma$. Using the Gluing Lemma 4.4 we get a simplex $w_\sigma \in \left(cl^{n-q} C_*(L_\sigma; \mathbb{Z}_2)\right)_{k+1}$ satisfying (4.6). $\square$

The proof above exhibits a close relationship between 1-dimensional quantities and fundamental classes and calls to mind the statement and proof of the systolic inequality.

5 Essential manifolds

There is a natural generalisation of Theorem 2.1 to essential source manifolds M. We will recall this notion.

Definition 5.1 (Essentialness, cf. [Gro83])**.** Let G be an abelian coefficient group and M^n be a closed connected G-oriented manifold with fundamental group $\pi_1(M) =: \pi$ and fundamental class $[M]_G \in H_n(M; G)$. Let $\Phi\colon M \to B\pi$ denote the classifying map of the universal cover $\widetilde{M} \to M$. The manifold M is said to be G*-essential* if the image

$$\begin{aligned} \Phi_*\colon H_n(M;G) &\to H_n(B\pi;G) = H_n(\pi;G) \\ [M]_G &\mapsto \Phi_*[M]_G \neq 0 \end{aligned}$$

does not vanish.

Theorem 5.2. Let M^m be a manifold with fundamental group $\mathbb{Z}^n$ and assume that at least one of the following properties holds:

(i) M is $\mathbb{Z}_2$-essential

(ii) M and N are orientable and M is $\mathbb{Z}$-essential

Then every continuous map $f\colon M \to N$ admits a point $y \in N$ such that the rank of the restriction homomorphism satisfies

$$\operatorname{rk}\left[H^1(M;\mathbb{Z}) \to H^1(f^{-1}y;\mathbb{Z})\right] \geq m - q.$$

Remark 5.3. (i) With the assumptions of the theorem above we automatically have $m \leq n$ since $H_{>n}(B\mathbb{Z}^n;G) = H_{>n}(T^n;G) = 0$. Examples of G-essential n-manifolds with fundamental group $\mathbb{Z}^n$ ($m = n$) that are not necessarily tori are connected sums of T^n with any simply connected manifold in dimensions $n \geq 3$. If $4 \leq m < n$ we can start with a map $\varphi\colon T^m \to T^n$ such that $H_m(\varphi;G)[T^m] \neq 0$ and use surgery to turn this into an essential m-manifold with fundamental group $\mathbb{Z}^n$.

(ii) For orientable manifolds M^m with fundamental group $\mathbb{Z}^n$ and classifying map $\Phi\colon M \to T^n$ we have the following commutative diagram.

$$\begin{array}{ccccc} H_m(M;\mathbb{Z}) & \xrightarrow{H_m(\Phi;\mathbb{Z})} & H_m(T^n;\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^{\binom{n}{m}} \\ \downarrow & & \downarrow & & \downarrow \\ H_m(M;\mathbb{Z}_2) & \xrightarrow[H_m(\Phi;\mathbb{Z}_2)]{} & H_m(T^n;\mathbb{Z}_2) & \xrightarrow[\cong]{} & \mathbb{Z}_2^{\binom{n}{m}} \end{array}$$

The vertical arrows are change-of-coefficient homomorphisms and the leftmost vertical arrow maps $[M]_{\mathbb{Z}}$ to $[M]_{\mathbb{Z}_2}$. This diagram shows that for such manifolds $\mathbb{Z}_2$-essentialness implies $\mathbb{Z}$-essentialness. This explains the somehow inorganic essentialness assumption in the theorem above.

Proof of Theorem 5.2. We only discuss case (i) and indicate the necessary adaptions to the existing proof. Like we reduced Theorem 2.1 to Proposition 2.3 we proceed by contradiction and assume that N is connected and closed, there exists a smooth $f\colon M \to N$ and a triangulation $\mathcal{T}$ of N such that the following two properties hold:

(i) The smooth simplices of $\mathcal{T}$ intersect f stratum transversally.

(ii) For every $\sigma \in \mathcal{T}_k$ the inclusion $f_\sigma \colon F_\sigma := f^{-1}\sigma(\Delta^k) \hookrightarrow M$ satisfies

$$\operatorname{rk} H^1(f_\sigma;\mathbb{Z}) < m-q.$$

Again for every $\sigma \in \mathcal{T}_k$ we consider the simplices $z_\sigma \in (cl^{m-q}C_*(F_\sigma;\mathbb{Z}_2))_k$ and the canonical cycle $Z(f;\mathcal{T}) \in C_q cl^{m-q}C_*(M;\mathbb{Z}_2)$ from Construction 4.5. Every F_σ comes with a reference map to M and naïvely we would think that we are in need of a replacement for Filling Lemma 3.2 where all the maps have target M instead of T^n. Instead consider the classifying map $\Phi\colon M \to T^n$. The diagram

$$\begin{array}{ccccc}
Z(f;\mathcal{T}) & & \xrightarrow{\quad\Phi_*\quad} & & \Phi_* Z(f;\mathcal{T}) \\
\Big\downarrow{\scriptstyle \mathrm{ev}_q} & C_\bullet cl^{m-q}C_*(M;\mathbb{Z}_2) & \longrightarrow & C_\bullet cl^{m-q}C_*(T^n;\mathbb{Z}_2) & \Big\downarrow{\scriptstyle \mathrm{ev}_q} \\
 & \downarrow & & \downarrow & \\
 & C_{(m-q)+*}(M;\mathbb{Z}_2) & \longrightarrow & C_{(m-q)+*}(T^n;\mathbb{Z}_2) & \\
\widehat{Z(f;\mathcal{T})} & & \xrightarrow[\quad\Phi_*\quad]{} & & \Phi_*\widehat{Z(f;\mathcal{T})}
\end{array}$$

commutes, the bottom left cycle represents the fundamental class $[M]_{\mathbb{Z}_2} \in H_m(M;\mathbb{Z}_2)$ and since M is $\mathbb{Z}_2$-essential the bottom right cycle defines a non-zero element in $H_m(T^n;\mathbb{Z}_2)$. Therefore the top right cycle defines a non-zero element in $H_q cl^{m-q}C_*(T^n;\mathbb{Z}_2)$.

The map Φ induces an isomorphism on π_1 as well as on H_1 by the Hurewicz theorem and H^1 by Remark 3.3 (ii) (both with coefficients in $\mathbb{Z}$). This proves that every map $k\colon K \to T^n$ satisfying $\operatorname{rk} H^1(k;\mathbb{Z}) < n-q$ also satisfies

$$\operatorname{rk} H^1(\Phi\circ k;\mathbb{Z}) = \operatorname{rk} H^1(k;\mathbb{Z}) < n-q.$$

Hence we can proceed as earlier and deduce a contradiction by constructing a cone of $\Phi_* Z(f;\mathcal{T})$ in $cl^{m-q}C_*(T^n;\mathbb{Z}_2)$ via simplices $w_\sigma \in (cl^{n-q}C_*(T^n;\mathbb{Z}_2))_{k+1}$ satisfying

$$\partial_i w_\sigma = \begin{cases} w_{\partial_i\sigma} & \text{if } 0 \le i \le k \\ z_\sigma & \text{if } i = k+1. \end{cases}$$

For $\sigma \in \mathcal{T}_0$ and $i=0$ the equation above shall be interpreted as $\partial_0 w_\sigma = w_{\partial_0\sigma} = 0$. □

Question 5.4. (i) Theorem 2.1, the more general Theorem 5.2 and the core input of both, Filling Lemma 3.2, give the impression that we have not proven something about tori but about the geometry of the group $\mathbb{Z}^n$. Are there analogues for other groups G? Even in the case where G is abelian with torsion, this is harder because $B\mathbb{Z}_p$ has cohomology classes in arbitrary high degrees and not every cohomology class in H^*G is a product of degree 1 classes, although admittedly we only used this property in Motivation 3.1.

(ii) Michał Marcinkowski has asked whether Theorem 5.2 fails if M has fundamental group $\mathbb{Z}^n$ but is inessential.

6 Cartesian powers of higher–dimensional spheres, rational homotopy theory

There is another natural generalisation of Theorem 2.1 from tori to cartesian powers of higher-dimensional spheres. Our previous proof of Filling Lemma 3.2 used covering space theory and cannot be generalised to simply connected manifolds. Instead we will use rational homotopy theory.

Theorem 6.1. Let $p \geq 3$ be odd and $n \leq p-2$. Consider $M = (S^p)^n$ or any simply connected, closed manifold of dimension pn with the rational homotopy type $(S^p)^n_{\mathbb{Q}}$ and N^q an arbitrary orientable q-manifold. Every continuous map $f\colon M \to N$ admits a point $y \in N$ such that the rank of the restriction homomorphism satisfies

$$\operatorname{rk}\left[H^p(M;\mathbb{Q}) \to H^p(f^{-1}y;\mathbb{Q})\right] \geq n-q.$$

Remark 6.2. Examples of manifolds M as above that are not $(S^p)^n$ are products of rational homology spheres of dimension p or connected sums of $(S^p)^n$ with rational homology spheres of dimension pn.

In this section the coefficient ring is always $R = \mathbb{Q}$. We assume that the reader already got a rough idea of rational homotopy theory but before we prove the theorem above we will shortly recap the notions and concepts we are going to need (cf. [FHT05] and [FOT08]).

Definition 6.3 (Rationalisations)**.** For a map $f\colon X \to Y$ between simply connected spaces the following three conditions are equivalent:

(i) $\pi_* f \otimes \mathbb{Q}\colon \pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q}$ are isomorphisms

(ii) $H_*(f;\mathbb{Q})$ are isomorphisms

(iii) $H^*(f;\mathbb{Q})$ are isomorphisms

In this case f is called a *rational homotopy equivalence* which is denoted by

$$X \xrightarrow[\cong_{\mathbb{Q}}]{f} Y.$$

A space X is called *rational* if it is simply connected and all $\pi_* X$ are rational $\mathbb{Q}$-vector spaces. A rational homotopy equivalence between rational spaces is a homotopy equivalence.

For any simply connected X there exists a rational space $X_{\mathbb{Q}}$ and a continuous map $r_X\colon X \to X_{\mathbb{Q}}$ which is a rational homotopy equivalence. The space $X_{\mathbb{Q}}$ is called the *rationalisation of* X and r_X the *rationalisation map of* X. With these properties the homotopy type of $X_{\mathbb{Q}}$ is uniquely determined and is called the *rational homotopy type of* X.

Definition 6.4 (Piecewise polynomial differential forms)**.** To any topological space X we can associate a *commutative* differential graded algebra (henceforth abbreviated by *cgda*) $A_{PL}(X) := A_{PL}(X;\mathbb{Q})$. This cgda is called the algebra of *piecewise polynomial differential forms on* X and by definition an element $\omega \in A^k_{PL}(X)$ assigns to every singular n-simplex in X a *polynomial* degree k differential form on the standard n-simplex, consistent with face and degeneracy maps. This yields a contravariant functor $A_{PL}\colon \mathbf{sSet} \to \mathbf{cgda}$ and there is a natural isomorphism

$$H^* A_{PL}(X) \cong H^*(X;\mathbb{Q}). \tag{6.1}$$

Definition 6.5 (Sullivan and minimal algebras, minimal models)**.** A *Sullivan algebra* is a cdga $(\bigwedge V, d)$ whose underlying algebra is free commutative for some graded $\mathbb{Q}$-vector space $V = \bigoplus_{n\geq 1} V^n$ and such that V admits a basis (x_α) indexed by a well-ordered set such that $dx_\alpha \in \bigwedge(x_\beta)_{\beta<\alpha}$. It is called a *minimal algebra* if it satisfies the additional property $d(V) \subseteq \bigwedge^{\geq 2} V$.

A morphism of cgdas is called a *quasi-isomorphism* if it induces isomorphisms on all cohomology groups. A quasi-isomorphism

$$\left(\bigwedge V, d\right) \to (A, d)$$

from a minimal algebra to an arbitrary cgda (A, d) is called a *minimal model* of (A, d). If X is a topological space any minimal model

$$\left(\bigwedge V, d\right) \to A_{PL}(X)$$

is called a *minimal model of* X.

Every simply connected space admits such a minimal model. For any simply connected X the maps $H^*(r_X;\mathbb{Q})$ are isomorphisms and using (6.1) we conclude that $A_{PL}(r_X)$ is a quasi-isomorphism. If $m\colon (\bigwedge V, d) \to A_{PL}X_{\mathbb{Q}}$ is a minimal model of $X_{\mathbb{Q}}$ the composition

$$\left(\bigwedge V, d\right) \xrightarrow{m} A_{PL}X_{\mathbb{Q}} \overset{A_{PL}(r_X)}{\longrightarrow} X$$

yields a minimal model for X.

Example 6.6 (Minimal models of spheres, products)**.** (i) For the spheres $S^p_{\mathbb{Q}}$ we can give explicit models depending on the parity of p. If p is odd one particular model is given by

$$\left(\bigwedge[x], 0\right) \to A_{PL}S^p_{\mathbb{Q}}$$

with $\deg x = p$ and $d = 0$. If p is even there is a model

$$\left(\bigwedge[x,y],d\right) \to A_{PL}S^p_{\mathbb{Q}}$$

with $\deg x = p$, $\deg y = 2p-1$, $dx = 0$ and $dy = x^2$.

(ii) If $(\bigwedge V, d) \to A_{PL}X$ is a minimal model for X and $(\bigwedge W, d) \to A_{PL}Y$ one for Y then

$$\left(\bigwedge[V \oplus W], d\right) \cong \left(\bigwedge V, d\right) \otimes \left(\bigwedge W, d\right)$$

is a minimal model for the product $X \times Y$.

Definition 6.7 (Spatial realisation)**.** There is another contravariant functor $|\cdot|\colon$ **cgda** $\to$ **Top**, called *spatial realisation*, and for every space X a continuous map

$$h_X\colon X \to |A_{PL}(X)|.$$

These map are called *unit maps* and they are natural in X, i.e. for any continuous map $f\colon X \to Y$ the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ |A_{PL}X| & \xrightarrow[|A_{PL}f|]{} & A_{PL}Y \end{array}$$

commutes.

Theorem 6.8. The unit maps h_X are always *rational homology equivalences*, i.e. $H_*(h_X;\mathbb{Q})$ (or equivalently $H^*(h_X;\mathbb{Q})$) are isomorphisms. For any rational space $X_{\mathbb{Q}}$ and any minimal model $m\colon\ (\bigwedge V, d) \to A_{PL}X_{\mathbb{Q}}$ the maps

$$h_{X_{\mathbb{Q}}}\colon X_{\mathbb{Q}} \xrightarrow{\simeq} |A_{PL}X_{\mathbb{Q}}|$$

and

$$|m|\colon |A_{PL}X_{\mathbb{Q}}| \xrightarrow{\simeq} \left|\bigwedge V, d\right|$$

are homotopy equivalences.

Now we can start proving Theorem 6.1. As a natural replacement for Motivation 3.1 we have the following

Lemma 6.9. Let $(\bigwedge[x_1,\ldots,x_n],0)$ be the minimal cgda with all generators concentrated in degree p and (A,d) an arbitrary cgda. Any morphism

$$f^\sharp\colon\ \left(\bigwedge[x_1,\ldots,x_n],0\right) \to (A,d)$$

with $\operatorname{rk} H^p(f^\sharp) < n-q$ satisfies $H^{\geq(n-q)p}(f^\sharp) = 0$.

Proof. The statement is non-vacuous only in degrees $lp \geq (n-q)p$, i.e. $l \geq n-q > \operatorname{rk} H^p(f^\sharp)$. From now on the proof is the same as that of Motivation 3.1. $\square$

Lemma 6.10. Let $n \leq p-2$. For any $0 \leq a < n$ the linear diophantine equation

$$\lambda(p-1) + \mu p = np - a \tag{6.2}$$

has exactly one solution $(\lambda, \mu) \in \mathbb{Z}^2_{\geq 0}$ given by $(\lambda, \mu) = (a, n-a)$.

Proof. The integer solutions of (6.2) are parametrised by

$$\{(\lambda, \mu) = (a + kp, (n-a) - k(p-1)) | k \in \mathbb{Z}\}.$$

Then the additional requirement $\lambda, \mu \geq 0$ translates into

$$-\frac{a}{p} \leq k \leq \frac{n-a}{p-1}. \tag{6.3}$$

Since

$$\frac{n-a}{p-1} - \left(-\frac{a}{p}\right) \leq \frac{n-a}{p-1} + \frac{a}{p-1} = \frac{n}{p-1} < 1$$

inequality (6.3) has at most one solution. It is easy to check that $(a, n-a)$ satisfies all desired properties. $\square$

The lemma above will enable us to prove the following rational version of Filling Lemma 3.2.

Lemma 6.11 (Rational Filling Lemma)**.** Let $p \geq 3$ be odd, $n \leq p-2$, $q < n$ and $k \colon K \to (S^p)^n_{\mathbb{Q}}$ a continuous map with $H^p(k; \mathbb{Q}) < n - q$. There exists a relative CW complex $(\operatorname{Fill}(k), K)$ and an extension $\operatorname{fill}(k) \colon \operatorname{Fill}(k) \to (S^p)^n_{\mathbb{Q}}$ such that the diagram

$$\begin{array}{ccc} \operatorname{Fill}(k) & & \\ \uparrow{\scriptstyle \iota} & \searrow^{\operatorname{fill}(k)} & \\ K & \xrightarrow[k]{} & (S^p)^n_{\mathbb{Q}} \end{array}$$

commutes and the following properties hold.

(i) $H_{\geq np-q}(\iota; \mathbb{Q}) = 0$

(ii) $H^p(\operatorname{Fill}(k), K; \mathbb{Q}) = 0$

(iii) $\operatorname{rk} H^p(\operatorname{fill}(k); \mathbb{Q}) = H^p(k; \mathbb{Q}) < n - q$

Proof. The proof strategy is to solve the problem on the algebraic level of cgdas and then use spatial realisation to obtain the desired spaces and maps. Since p is odd we have a minimal model

$$\left(\bigwedge[x_1, \ldots, x_n], 0\right) \to A_{PL}(S^p)^n_{\mathbb{Q}}$$

with generators x_i concentrated in degree p (cf. Example 6.6). Consider $k^\sharp$ given by the following diagram.

$$\begin{array}{ccc} A_{PL}K & \xleftarrow{A_{PL}k} & A_{PL}(S^p)^n_{\mathbb{Q}} \\ & {}^{k^\sharp}\nwarrow & \uparrow \\ & & (\bigwedge[x_1,\ldots,x_n],0) \end{array} \tag{6.4}$$

Morphisms between cdgas are denoted with a lower case letter endowed with the superindex $\sharp$. This notation shall hint at which continuous map we will get after spatial realisation. The map $k^\sharp$ can be factored as follows.

$$\begin{array}{ccc} A_{PL}K & & \\ {\scriptstyle \iota^\sharp}\uparrow & \overset{k^\sharp}{\nwarrow} & \\ \left(\bigwedge\left[H^{p-1}K \oplus \operatorname{im} H^p(k^\sharp)\right],0\right) & \xleftarrow[g^\sharp]{} & (\bigwedge[x_1,\ldots,x_n],0) \end{array} \tag{6.5}$$

The morphism $g^\sharp$ is the obvious one. The map $\iota^\sharp$ can be defined by *choosing representing cocycles*, i.e. choose $y_i \in A_{PL}^{p-1}K$ such that $[y_i]$ constitutes a basis of $H^{p-1}(A_{PL}K)$ and define

$$\begin{aligned} \iota^\sharp\colon \left(\bigwedge\left[H^{p-1}K \oplus \operatorname{im} H^p(k^\sharp)\right],0\right) &\to A_{PL}K \\ [y_i] &\mapsto y_i \\ H^p(k^\sharp)[x_i] &\mapsto k^\sharp x_i. \end{aligned}$$

With this definition $H^{p-1}\iota^\sharp$ is injective and $H^p\iota^\sharp$ is surjective both of which will in due course imply (ii) and (iii).

We are left to prove (i) which is equivalent to $H^{\geq pn-q}\iota^\sharp = 0$. For $0 \leq a \leq q < n$ consider a degree $pn-a$ element $x \in \bigwedge\left[H^{p-1}K \oplus \operatorname{im} H^p(k^\sharp)\right]$. We will show that $H^{pn-a}\iota^\sharp[x] = 0 \in H^pA_{PL}K$. Without loss of generality x is a product of λ generators of degree $(p-1)$ and μ generators of degree p. Since $n \leq p-2$ Lemma 6.10 yields $(\lambda,\mu) = (a, n-a)$. Thus x contains at least $n-q$ generators of degree p, i.e.

$$x = yz_1 \cdot \ldots \cdot z_{n-q}$$

and the z_i can be written as $z_i = g^\sharp w_i$ for some $w_i \in [x_1,\ldots,x_n]$. We conclude

$$\begin{aligned} H^{pn-a}\iota^\sharp[x] &= [\iota^\sharp x] = [\iota^\sharp(yz_1 \cdot \ldots \cdot z_{n-q})] = [(\iota^\sharp y)(\iota^\sharp z_1) \cdot \ldots \cdot (\iota^\sharp z_{n-q})] \\ &= [(\iota^\sharp y)(\iota^\sharp g^\sharp w_1) \cdot \ldots \cdot (\iota^\sharp g^\sharp w_{n-q})] = [\iota^\sharp y] H^{(n-q)p} k^\sharp [w_1 \cdot \ldots w_{n-q}]. \end{aligned}$$

Using the natural isomorphism (6.1) we get that $\operatorname{rk} H^p(k^\sharp) < n-q$. Hence we can apply the preceding lemma to conclude $H^{\geq(n-q)p}(k^\sharp) = 0$ proving $H^{pn-a}\iota^\sharp[x] = 0$.

Let

$$\left(\bigwedge W, 0\right) := \left(\bigwedge \left[H^{p-1}K \oplus \operatorname{im} H^p(k^\sharp)\right], 0\right).$$

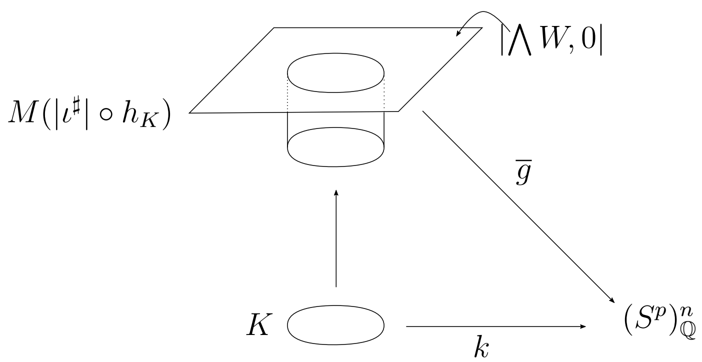
After spatial realisation of diagrams (6.4) and (6.5) and introducing the unit maps from Definition 6.7 we get the following diagram.

$$\begin{array}{ccc} K & \xrightarrow{\quad k \quad} & (S^p)^n_{\mathbb{Q}} \\ {\scriptstyle h_K}\big\downarrow & & \big\downarrow {\scriptstyle h_{(S^p)^n_{\mathbb{Q}}}}\ \simeq \\ |A_{PL}K| & \longrightarrow & \left|A_{PL}(S^p)^n_{\mathbb{Q}}\right| \\ {\scriptstyle |\iota^\sharp|}\big\downarrow & & \big\downarrow \simeq \\ |\bigwedge W, 0| & \xrightarrow[|g^\sharp|]{} & |\bigwedge[x_1, \ldots, x_n], 0| \end{array}$$

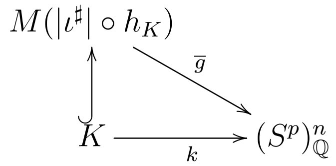
In this diagram the upper square commutes strictly but the lower one only up to homotopy (cf. Definition 6.7). By Theorem 6.8 the map h_K is a rational cohomology equivalence, in particular we still have that $H^{p-1}(|\iota^\sharp| \circ h_K)$ is surjective and $H^p(|\iota^\sharp| \circ h_K)$ is injective. The same theorem states that the right hand side vertical arrows are homotopy equivalences. After choosing homotopy inverses we get the triangle

$$\begin{array}{ccc} K & \xrightarrow{\ k\ } & (S^p)^n_{\mathbb{Q}} \\ {\scriptstyle |\iota^\sharp| \circ h_K}\big\downarrow & \nearrow {\scriptstyle \widehat{g}} & \\ |\bigwedge W, 0| & & \end{array}$$

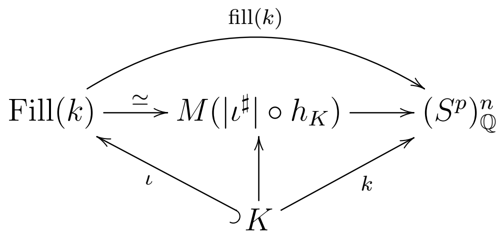
which commutes up to homotopy. Choose such a homotopy $H\colon \widehat{g} \circ (|\iota^\sharp| \circ h_K) \simeq k$ and consider the mapping cylinder of $|\iota^\sharp \circ h_K|$.



Using the homotopy H we get a map $\overline{g}$ such that the diagram



commutes strictly. Choose a relative CW approximation $(\mathrm{Fill}(k), K) \to (M(|\iota^\sharp| \circ h_K), K)$, i.e. a relative CW complex $(\mathrm{Fill}(k), K)$ together with a map $\mathrm{Fill}(k) \to M(|\iota^\sharp| \circ h_K)$ which is a homotopy equivalence and restricts to the identity on K. Define ι and $\mathrm{fill}(k)$ as in the following diagram.



The induced map $H^{p-1}(\iota; \mathbb{Q})$ is surjective and $H^p(\iota; \mathbb{Q})$ is injective since $|\iota^\sharp| \circ h_K$ has these properties. From this we get $H^p(\mathrm{Fill}(k), K; \mathbb{Q}) = 0$ hence $H_p(\mathrm{Fill}(k), K; \mathbb{Q}) = 0$. As usual we successively conclude that $H_p(\iota; \mathbb{Q})$ is surjective and $\mathrm{rk}\, H^p(\mathrm{fill}(k); \mathbb{Q}) = \mathrm{rk}\, H^p(k; \mathbb{Q})$. □

Remark 6.12. (i) In the case $p = 1$ the factorisation (6.5) reminds us of our original Filling Lemma 3.2.

(ii) The condition $n \leq p - 2$ seems a little inorganic. But in the case $n = p - 1$ the element x could be of degree np and therefore the product of p generators of degree $(p-1)$ and we would not have any control over the image $H^{np}\iota^\sharp[x]$. We do not know how to weaken this condition. This may be possible by altering the construction of Rational Filling Lemma 6.11.

(iii) If p is even a minimal model of $(S^p)^n$ is given by $(\bigwedge[x_1, \ldots, x_n, y_1, \ldots, y_n], d)$ with $dy_i = x_i^2$. However it is not clear what the image of y_i under the map $g^\sharp$ should be such that diagram (6.5) commutes or how to alter the construction.

(iv) It is remarkable that Rational Filling Lemma 6.11 can be proven while almost exclusively manipulating algebraic objects.

Proof of Theorem 6.1. We will only indicate how to change the existing proof scheme. Again we proceed by contradiction and assume that N connected and closed, there exists a smooth $f\colon M^{np} \to N^q$ and a triangulation $\mathcal{T}$ of N such that the following two properties hold:

(i) The smooth simplices of $\mathcal{T}$ intersect f stratum transversally.

(ii) For every $\sigma \in \mathcal{T}_k$ the inclusion $f_\sigma \colon F_\sigma := f^{-1}\sigma(\Delta^k) \hookrightarrow M$ satisfies

$$\operatorname{rk} H^p(f_\sigma; \mathbb{Q}) < n - q.$$

Again for every $\sigma \in \mathcal{T}_k$ we consider the simplices $z_\sigma \in (cl^{np-q}C_*(F_\sigma; \mathbb{Q}))_k$ and the canonical cycle $Z(f; \mathcal{T}) \in C_q cl^{np-q} C_*(M; \mathbb{Q})$ from Construction 4.5. Let $r_M \colon M \to (S^p)^n_{\mathbb{Q}}$ be the rationalisation map of M. The diagram

$$\begin{array}{ccccc}
Z(f;\mathcal{T}) & & \xrightarrow{\quad (r_M)_* \quad} & & (r_M)_* Z(f;\mathcal{T}) \\
\Big\downarrow & & & & \Big\downarrow \\
\mathrm{ev}_q & C_\bullet cl^{np-q} C_*(M;\mathbb{Q}) & \longrightarrow & C_\bullet cl^{np-q} C_*((S^p)^n_{\mathbb{Q}};\mathbb{Q}) & \mathrm{ev}_q \\
 & \downarrow & & \downarrow & \\
 & C_{(np-q)+*}(M;\mathbb{Q}) & \longrightarrow & C_{(np-q)+*}((S^p)^n_{\mathbb{Q}};\mathbb{Q}) & \\
\Big\downarrow & & & & \Big\downarrow \\
\widehat{Z(f;\mathcal{T})} & & \xrightarrow[\quad (r_M)_* \quad]{} & & (r_M)_* \widehat{Z(f;\mathcal{T})}
\end{array}$$

commutes. The bottom left cycle represents the fundamental class $[M]_{\mathbb{Q}} \in H^{np}(M; \mathbb{Q})$ and by Definition 6.3 r_M is a rational homology equivalence, in particular the bottom right cycle defines a non-zero element in $H_{np}((S^p)^n_{\mathbb{Q}}; \mathbb{Q})$. Therefore the top right cycle defines a non-zero element in $H_q cl^{np-q} C_*((S^p)^n_{\mathbb{Q}}; \mathbb{Q})$.

Now we can use the Rational Filling Lemma 6.11, proceed as earlier and deduce a contradiction by constructing a cone of $(r_M)_* Z(f; \mathcal{T})$ in $cl^{np-q}C_*((S^p)^n_{\mathbb{Q}}; \mathbb{Q})$ via simplices $w_\sigma \in \big(cl^{np-q}C_*((S^p)^n_{\mathbb{Q}}; \mathbb{Q})\big)_{k+1}$ satisfying

$$\partial_i w_\sigma = \begin{cases} w_{\partial_i \sigma} & \text{if } 0 \le i \le k \\ z_\sigma & \text{if } i = k+1. \end{cases}$$

For $\sigma \in \mathcal{T}_0$ and $i = 0$ the equation above shall be interpreted as $\partial_0 w_\sigma = w_{\partial_0 \sigma} = 0$. $\square$

5 Subsumption

All the previous work is almost exclusively inspired by perspectives from [Gro09] and [Gro10]. In this chapter we render this relation more precisely.

In [Gro10, section 4.13 Perspectives and problems, p. 521] the following programme called "Homological filling" was proposed.

> Eventually, we want to find lower bounds on the cohomological $\mathrm{width}^*(X/Y)$, say for $Y = \mathbb{R}^m$, by a filling argument similar to that in 2.4, but one needs for this, besides filling inequalities, an appropriate semisimplicial structure in the space of cycles in $A = H^*(X)$. It seems unlikely, however, that this structure can be constructed while remaining in within $H^*(X)$, since "gluing fillings across common boundaries" involves (the multiplicative structure on) the relative cohomology that is not contained in the restriction homomorphism alone. In any case, a realistic evaluation of the cohomological $\mathrm{width}^n(X/\mathbb{R}^m)$ remains open even for such X as the product of Eilenberg–MacLane spaces.

In Chapter 4 we have found such cohomological waist inequalities using the aforesaid filling argument (cf. Proposition 2.3, Theorem 5.2 and Theorem 6.1) and cohomological filling inequalities. We propose the following rigorous

Definition 1.1 (Cohomological filling inequality)**.** Let X be a topological space, $p, m \geq 1$ and let R be a coefficient ring such that the rank of a morphism of R-modules makes sense, e.g. $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Q}$. A function

$$\mathcal{F}\colon \{0, \ldots, |H^k X|\} \to \{0, \ldots, |H^k X|\}$$

is called a *cohomological filling inequality for X in dimension m and degree k* if for any continuous map $k\colon K \to X$ and every homology class $[K] \in H_m(K)$ with $H_m(k)[K] = 0$ there exists a relative CW complex $(\mathrm{Fill}(k), K)$ together with an extension

$$\begin{array}{ccc} \mathrm{Fill}(k) & & \\ {\scriptstyle\iota}\uparrow & \searrow^{\mathrm{fill}(k)} & \\ K & \xrightarrow[k]{} & X \end{array} \tag{1.1}$$

satisfying the following two properties.

(i) $H_m(\iota)[K] = 0$

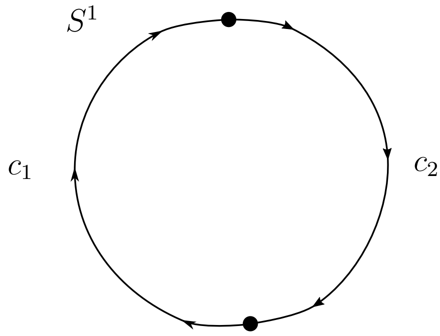
(ii) $\operatorname{rk} H^p \operatorname{fill}(k) \leq \mathcal{F}(\operatorname{rk} H^p k)$

Remark 1.2. (i) Property (i) is a witness for the assumption $k_*[K] = 0$. Of course one could take $\operatorname{Fill}(k) := X$, $\iota := k$ and $\operatorname{fill}(k) := \operatorname{id}_X$ but in this case $\operatorname{rk} H^k \operatorname{fill}(k) = \operatorname{rk} H^k X$ which is a very weak cohomological filling inequality. In this sense properties (i) and (ii) act against each other.

(ii) Since $\operatorname{rk} H^p(\operatorname{fill}(k)) \geq \operatorname{rk} H^p(k)$ it is clear that $\mathcal{F}(l) \geq l$ for every $0 \leq l \leq |H^p(X)|$.

(iii) With this terminology Filling Lemma 4.3.2 is a cohomological filling inequality for T^n in all dimensions $m \geq n - q$ and degree 1. The Rational Filling Lemma 4.6.11 works for all simply connected manifolds $X \simeq_{\mathbb{Q}} (S^p)^n$ in all dimensions $m \geq np - q$ and in degree p. In both cases we have proven the optimal case where $\mathcal{F}$ is the identity. Remark that both T^n and $(S^p)^n$ (p odd) are (rational) Eilenberg–MacLane spaces. It is plausible to search for cohomological filling inequalities in these since finding an extension like in diagram (1.1) is essentially a problem in obstruction theory which is very simple for Eilenberg–MacLane spaces.

Recall the notion of cohomological volume from Definition 1.4.5. Consider the following two chains $c_1, c_2 \in C_1(S^1; \mathbb{Z})$ the sum of which represents the fundamental class in $H_1(S^1; \mathbb{Z})$.



These two chains satisfy $|c_i|_1 = 0$ but we have $|c_1 + c_2|_1 = 1$. This shows that cohomological volume fails any kind of additivity axiom. As anticipated in the quote above this can be circumvented by using relative cohomology groups. In all of our cohomological filling inequalities we constructed a fillings $(\operatorname{Fill}(k), K)$ such that a certain relative homology group $H_*(\operatorname{Fill}(k), K)$ vanishes and this ensured (cf. the proof of Proposition 4.2.3) that we have control over this lack of additivity of cohomological volume.

As we already saw in Remark 4.3.5 (i) this trivial relative homology group guarantees that the reference map $\operatorname{Fill}(k) \to T^n$ is unique up to homotopy relative to K and this makes it easy to consistently glue together fillings across common boundaries.

In the same section as above [Gro10, pp. 520] we can find the following passage.

The simplest (co)homological model of an n-cycle c in a space X with cohomology algebra $A = H^*(X;\mathbb{F})$ is given by a graded algebra $C = C(c)$, a graded homomorphism $h = h(c)\colon A \to C$ and a linear map $l = l(c)\colon C \to \mathbb{F}$.

Denote,

$$|c|_{\mathbb{F}} = |C|_{\mathbb{F}}, \quad |A/c|_{\mathbb{F}} = \mathrm{rank}_{\mathbb{F}}(h) \quad \text{and} \quad [c] = l \circ h\colon A \to \mathbb{F}.$$

We think of c as a representative of the class $[c]$ and introduce the following "norms":

$$\inf_{c\in[c]} |c|_{\mathbb{F}} = |C(c)|_{\mathbb{F}} \quad \text{and} \quad |A/[c]|_{\mathbb{F}} = \inf_{c\in[c]} |A/c|_{\mathbb{F}}.$$

The first (apparently easy) question is the evaluation of these "norms" (ranks) on linear maps $[c]\colon A \to \mathbb{F}$ for particular algebras A, e.g. for the cohomology algebras A of products of Eilenberg–MacLane spaces.

Next, if a "cycle" c has $[c] = 0$, we define a "filling" b of c as an algebra B and a decomposition of $h\colon A \to C$ into homomorphisms $A \overset{g_1}{\to} B \overset{g_2}{\to} C$. Then we set

$$\|c\|_{\mathrm{fil}} = \inf_b |b|_{\mathbb{F}} = |B|_{\mathbb{F}} \quad \text{and} \quad \|A/c\|_{\mathrm{fil}} = \inf_b |A/b|_{\mathbb{F}} = \mathrm{rank}_{\mathbb{F}}(g_1)$$

where the infima are taken over all "fillings" b of c.

What are the "filling inequalities" between $|[c]|_{\mathbb{F}}$ and $|A/[c]|_{\mathbb{F}}$ on the one hand and their filling counterparts $\|c\|_{\mathrm{fil}}$ and $\|A/c\|_{\mathrm{fil}}$ on the other for particular algebras A (e.g. for free anticommutative algebras)?

Our exegesis of this quote is that the problem of finding cohomological fillings in the sense of Definition 1.1 already may have purely cohomological obstructions. On the other hand the proof of Rational Filling Lemma 4.6.11 showed that sometimes solving the problem on the purely algebraic level of minimal models suffices since it fully reflects the topological picture. Using this powerful feature of rational homotopy theory is multiply suggested, e.g. in the perspective "Homological filling" [Gro10, p. 521]:

If $\mathbb{F} = \mathbb{F}_p$, then A comes with an action of the Steenrod algebra and one may insist on C and h being compatible with this action and if $\mathbb{F} = \mathbb{Q}$ one may use the full minimal model of X instead of the cohomology.

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